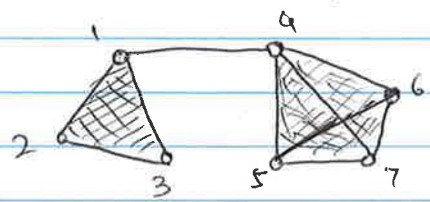
Jan 24

→ Last class: Simplex
→ Today: Simplicial Complex (SC)
• Types of Simplicial Complex:
  1. Abstract S.C
  2. Vietoris-Rips S.C (Rips complex) } saw in last class
  3. Čech Complex }
  4. Delaunay Complex [Closely related to Voronoi diagrams]
  5. Alpha Complex [Applications in protein docking]
  6. Witness Complex } sparsified SC
  7. Graph-induced Complex }

→ Delaunay Complex (triangulation) is the dual of Voronoi dgm.

→ Combinatorial structures we can impose on Point Cloud data (PCD)
(a) Graphs: describe pair-wise relations
  ↳ hypergraphs: edges / hyperedges connecting > 2 vertices

(b) Simplicial Complexes: describe higher order interaction

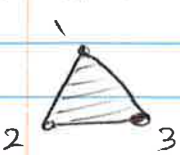


(1,2,3) → Graph only describes edges 12, 23, 13
SC → in addition to the three edges
The SC also includes the triangle 123.
Even higher order: for points (4,5,6,7) the SC includes all edges, all triangles and the solid tetrahedron.

Def: A k-simplex is the convex hull of $k+1$ affinely independent points. denote: $\sigma_k = \text{Conv}\{u_0, u_1, \dots, u_k\}$

Def: A face of simplex (σ_k) is the convex hull of a non-empty subset of the $k+1$ points.
→ it is proper face if subset ≠ entire set.

eg.

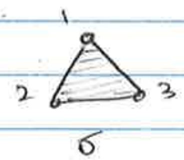


2-faces: 12, 23, 13
1-faces: 1, 2, 3
123 → Not proper

We denote Simplex by σ. Face of σ by τ

Def: If τ is the face of σ, then σ is Co-face of τ

Def: The boundary of simplex σ is the union of all the proper faces.

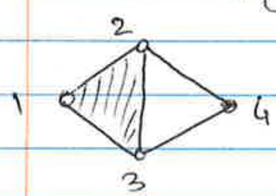


$bd(\sigma) = \bigcup$ 12, 23, 13, 1, 2, 3

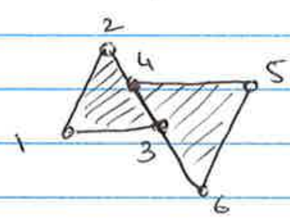
Def: An abstract SC is a finite Collection of sets A such that for any $\alpha \in A$, if $\beta \subseteq \alpha$ then $\beta \in A$.

Def: A simplicial complex K is a finite collection of simplexes such that:

1. for all simplexes $\sigma \in K$, if τ is a face of σ, $\tau \in K$
2. if $\sigma_1, \sigma_2 \in K$ then their intersection is either empty or a face of both σ_1 and σ_2



$K = \{1, 2, 3, 4, 12, 23, 13, 24, 34, 123\}$



$\rightarrow$ Not a SC. The intersection of two triangles is Not an edge of either (② Not satisfied)

Def: $|K|$ is the underlying space of SC K. it's the union of simplices in K together with the topology of the ambient Euclidean space those simplices live in.

Def: $L \subseteq K$ A subcomplex of SC K is a simplicial Complex L which is subset of K.

Def: A j-skeleton of K is the subset of all simplices in K of dimension $\leq j$

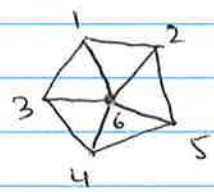
$$K^{(j)} = \{\sigma \in K \mid \dim(\sigma) \leq j\}$$

→ 0-skeleton : point cloud → Vertex (K)
1- Skeleton : Graph → (Vertex (K) , edges (K))

→ local neighborhood of a vertex in a graph is the set of vertices adjacent to it (reach in 1 hop)
We can extend the definition (vertices reached in 2 hops)

<u>Def</u> : Star of τ (denote $st(\tau)$) is the collection of all Co-faces of τ.

$$st(\tau) = \{ \sigma \in K \mid \underbrace{\tau \leq \sigma}_{\text{all } \sigma \text{ s.t. } \tau \text{ is a face of } \sigma} \}$$



$st(\{6\})$: All simplexes that have vertex 6 as a face.

→ $\{ 16, 26, 36, 46, 56, 136, 346, 456, 526, 126, 6 \}$
i.e. all edges & triangles containing 6 and the vertex 6 itself

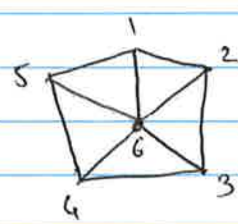
<u>Def</u> : $\overline{st}(\tau)$: Closed star of τ is the smallest subcomplex of K that contains the star of τ.

eg. from above SC, $\overline{st}(\tau) = st(\tau) \cup \{ 12, 13, 34, 45, 25, 1, 2, 3, 4, 5 \}$

<u>Def</u> : Link of τ (denote $Lk(\tau)$) is the set of simplexes in the closed star of τ but not in star of τ

$$Lk(\tau) = \{ \sigma \in \overline{st}(\tau) \mid \sigma \cap \tau = \emptyset \}$$

$st(123) = \{ 1235, 1234 \}$



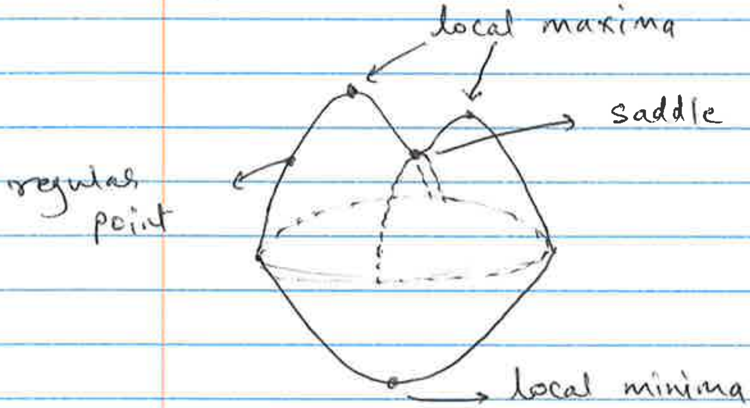
$st(6) = \{ 16, 26, 36, 46, 56, 6$
$126, 236, 346, 456, 156 \}$

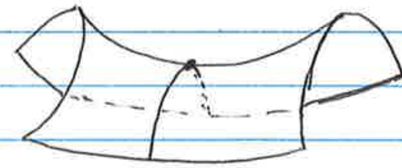
$\overline{st}(6) = st(6) \cup \{ 12, 23, 34,$
$1, 2, 3, 4, 5 \leftarrow 45, 51 \}$

$Lk(6) = \{ 12, 23, 34, 45, 51$
$1, 2, 3, 4, 5 \}$

→ Suppose we have a simplicial complex and a function defined on it

eg. geographical data and the function is elevation.





→ Link and star of a vertex can be used to decide the type of critical points in the terrain.

Def! $Lk_{-}(u) = \{ \sigma \in Lk(u) \mid x \in \sigma,\ f(x) < f(u) \}$

$Lk_{-}(u) \rightarrow$ marked edges & all other vertices <u>if</u> u is the local maxima.