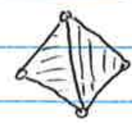
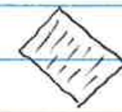
Feb 2

Review: Simplicial Complex K : ~~Co~~ Collection of simplices such that ~~face~~ if $\sigma \in K$, face of $\sigma \in K$ and for $\sigma_1, \sigma_2 \in K$, their intersection is a simplex $\rightarrow$ also belongs to K

Underlying space of $|K|$:



K



$|K|$ underlying topological space.

Def: A triangulation of a topological space $\mathbb{X}$ is a simplicial complex K together with a homeomorphism bet^n $\mathbb{X}$ & $|K|$

$\rightarrow$ Underlying space of triangulation (sc K) is homeomorphic to $\mathbb{X}$

Betti Numbers: (Homology in a nutshell)

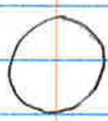
β_0 or b_0 : # of connected components

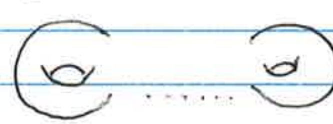
β_1 or b_1 : # of tunnels / loops

β_2 : # of voids

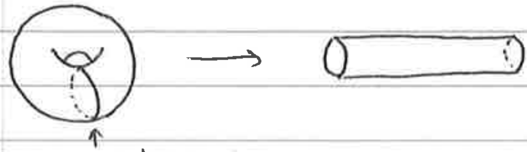
β_K : # of higher order voids

\# β_K : Rank of the K^{th} homology group (rank H_K)

		β_0	β_1	β_2		β_0	β_1	β_2
S^1 Circle		1	1	0	$\mathbb{K}^2$ Klein bottle	1	1	0
S^2 Sphere		1	0	1	2 hole torus	1	4	1
$\mathbb{T}^2$ Torus		1	2	1	g-hole torus (g → genus)	1	2g	1
$\mathbb{P}^2$ projective plane		1	0	0				

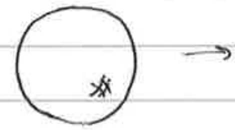


Def: The genus of a connected orientable surface is an integer representing the maximum number of cutting along simple closed curves without disconnecting the resulting manifold.



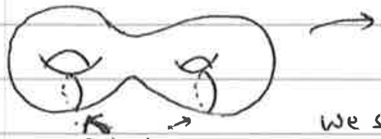
g_1

cut here



g_0

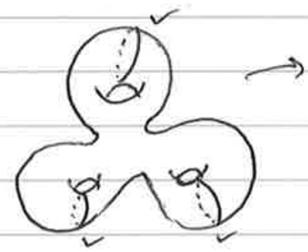
Sphere



g_2

Cut here

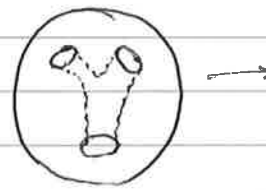
we still have connected manifold



g_3

g_ϕ

tube through sphere



g_2

Groups, Abelian groups

Def: A group is a set G with operation $\cdot$ such that

1. Closure: $a, b \in G$ then $a \cdot b \in G$
2. Associativity: $a, b, c \in G$ then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. identity: $\exists\, e \in G$ s.t. $e \cdot a = a \cdot e = a$ for any $a \in G$
4. inverse: $\forall\, a \in G\ \exists\, b \in G$ s.t. $a \cdot b = b \cdot a = e \rightarrow$ identity

satisfies 1 to 4

Abelian group is a group with additional property that

5. $a, b \in G$ then $a \cdot b = b \cdot a$

example: $(\mathbb{Z}, +)$: set of integers with addition operation

1. $a, b \in \mathbb{Z}$, $a + b \in \mathbb{Z}$
2. $a, b, c \in \mathbb{Z}$, $a + (b + c) = (a + b) + c$
3. identity is $0 \in \mathbb{Z}$ $\because a + 0 = 0 + a = a$
4. inverse is $-a$ $\forall\, a \in \mathbb{Z}$

   $a + -a = -a + a = 0$
5. Abelian $a, b \in \mathbb{Z}$, $a + b = b + a$

example $(\mathbb{Z}_2, +)$: integers modulo 2 with addition

$\mathbb{Z}_2 = \{0, 1\}$: ① $1+1 = 0$ ($2 \bmod 2$)

$0+1 = 1+0 = 1 \in \mathbb{Z}$

② identity is 0, ③ inverse is element itself

$0+0 = 0$, $1+1 = 0$

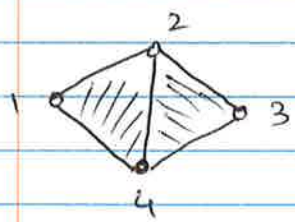
This is an Abelian group.

Homology: Let K be a simplicial complex, ~~of~~ denote dimension $= p$

Def: modulo 2 coefficient ($\mathbb{Z}_2$ coeff.) $a \in \mathbb{Z}_2$ $a = 0$ or $a = 1$

Def: A p-chain is a formal sum of p-simplices in K

$$c = \sum_i a_i \sigma_i \quad \text{where } a_i \in \mathbb{Z}_2 \ (0 \text{ or } 1)$$



$K = \{1, 2, 3, 12, 14, 24, 23, 34, 124, 234\}$

c_0 : 0-chain $=$ $1+2+3+4$, $1+4$, $2+4$

c_1 : 1-chain $= 12 + 23 + 34 + 41$

c_2 : 2-chain $= 124 + 234$

→ Let $c_0 = 1+2+3$, $c_0' = 1+3+4$ then $c_0 + c_0' = 1+2+3+1+3+4$
$= 2+4$ (modulo 2)

→ Let $c_1 = 12 + 23 + 34 + 14$

$c_1' = 23 + 34 + 24$

$c_1 + c_1' = 12 + 24 + 14$

→ Chain addition : Component-wise addition modulo 2

$c = \sum_i a_i \sigma_i$

$c' = \sum b_i \sigma_i$

$\Rightarrow c + c' = \sum_i (a_i + b_i) \sigma_i$

modulo 2 coefficients : $a_i + b_i \in \{0, 1\}$

<u>Def</u>: Chain group: The set of all p-chains with the addition operation form a group. $(C_p, +)$ or $C_p(K)$

C_0: 0-chain group $\quad C_1$: 1-chain group $\quad C_2$: 2-chain group.

→ ① $c, c' \in C_p$, $c + c' \in C_p$

② $(c+c')+c'' = c+(c'+c'')$

③ $0 + c = c + 0 = c$

④ $c + c = 0$

★ Addition is component-wise sum modulo 2

i.e. coefficients $\in \{0,1\}$

<u>Def</u>: Boundary of a p-simplex is the sum of its ~~(p-1) faces~~ (p-1)-dimensional faces

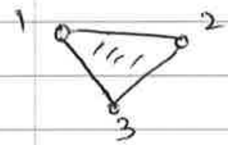
$$\sigma = [u_0, \dots, u_p],$$

$$\partial_p \sigma = \sum_{j=0}^{p} [u_0, \dots u_{j-1}, u_{j+1}, \dots u_p]$$

↓ u_j removed.

$$= \sum_{j=0}^{p} [u_0, \dots, \hat{u}_j, \dots u_p]$$

$\hat{u}_j \Rightarrow u_j$ removed.



$$\sigma = [1,2,3]$$

$$\partial_2 \sigma = \partial_2 [1,2,3]$$

$$= [\hat{1}, 2, 3] \to 23$$

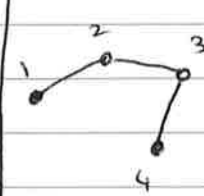
$$+ [1, \hat{2}, 3] \to 13$$

$$+ [1, 2, \hat{3}] \to 12$$

∴ Boundary of a triangle is sum of its edges

$c = [1,2]$

$\partial c = 1 + 2$

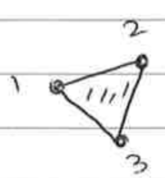


$c = 12 + 23 + 34$

$\partial c = 1 + 4$

→ Let $c = 12 + 23 + 34$

$$\partial c = 1+2+2+3+3+4 = 1+4$$



$c = 123$

$\partial c = 12 + 23 + 13$

$\partial(\partial c) = 1+2+2+3+1+3 = 0$

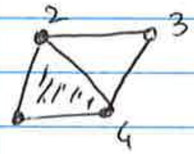
★ Boundary of a boundary is always 0.

Def: A p-cycle is a p-chain with empty boundary ($\partial c = 0$)

→ Set of all p-cycles $Z_p = Z_p(K)$ is a group that is subgroup of $C_p(K)$

Def: A p-boundary is a p-chain that is boundary of a (p+1)-chain $c = \partial d$ where $d \in C_{p+1}$

Set of all p-boundaries is a group $B_p = B_p(K)$



$c \in B_1(K)$

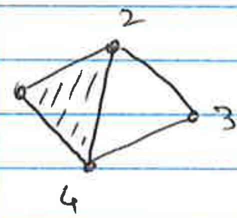
$c = \partial(124) = 12 + 24 + 14$

$c \in Z_1(K)$: $c = 23 + 34 + 24$, $c' = 12 + 24 + 14$

$c'' = 12 + 23 + 34 + 14$

Def: The p-th homology group is the p-th cycle group modulo the p-th boundary group $H_p = Z_p / B_p$

"Cycles that don't bound" ⇒ p-cycle that is not boundary of any (p+1)-chain.



E (23 + 34 + 24) and (12 + 23 + 34 + 14) belong to Z_1 but are not boundary of any 2-chain.

(12 + 24 + 14) → boundary of (124)