CS 6170 Computational Topology: Topological Data Analysis	**University of Utah**
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Lecture 17: Morse function - Mar 07, 2017

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17.1 Introduction - Types of analysis

1. Persistent Homology
   - Multi-Scala Data Analysis
2. Topological Data Structure: ($\mathbb{X}$: (high-dim) point set)
   - Scalar (Real-value) Function Analysis ($f : \mathbb{X} \to \mathbb{R}$)
   - Multivariate Function Analysis ($f : \mathbb{X} \to \mathbb{R}^d$)

We have spent the past week looking at persistent homology, and now we will look at some alternatives.

17.1.1 Scalar Function Analysis

One is scalar functions analysis, or sometimes more specifically real-valued functions analysis. And this arise quite a bit. For instance if you have nuclear stimulation or combustion stimulation. You have your point cloud, which for instance is the x, y, z coordinates of your actual stimulation or a multidimensional space. You always have a real valued function on the point cloud. Lets say $\mathbb{X}$ is potentially high dimensional point set, and function $f : \mathbb{X} \to \mathbb{R}$. Essentially every point in your data has a function attached to it. And this function value can be say in combustion stimulation could be temperature, or pressure. In another example, if you were doing GIS, which deals with terrain, every x, y point has elevation. The function is the height at the point. It's also referred as real valued function, where the function value at a location is a real value.

17.1.2 Multivariate Function Analysis

The other type I would call multivariate function analysis. There is 2 separate settings which are very close together. Multivariate essentially means that we have multiple observation at the same location. Once way you can think about it is I have a function on my domain, instead of having it be real valued, it can be vector valued. For example if I have a d dimension vector attached. So sometime I'm going to call this a vector valued function $f : \mathbb{X} \to \mathbb{R}^d$. Another way to think about this is d is how many observations I have at that location. I can refer to function $f_1, f_2, ..., f_d : \mathbb{X} \to \mathbb{R}$. I have d number of real valued functions in a real domain. Again, we can take combustion stimulation, at every x, y, z location, I can observe both temperature and pressure. So of course I can couple those values together to make it into a 2 dimensional vector, so my function is going to $\mathbb{R}^2$. The alternative is that f_1 is the temperature and f_2 is the pressure.

17.1.3 Techniques of Topological Data Structure

Given a real valued or multivariate function, how do we perform analysis with Topological Data Analysis techniques? Traditionally the technique used is regression. Given a data set, we typically attempt to find a model that fits. The techniques we will cover today are what I consider alternative or complimentary ways to analyze real valued or multivalued functions. What it boils down to is this. The number one tool in TDA is persistent homology. If we have a filtration describing the underlying space.

The idea is that if you have a filtration describing the evolution of the underlying space, I can use persistent homology to try to capture features that arise and disappear over multiple scale. In in some sense, persistent homology appears as a tool that understands data at multiple scale. For the remaining part we will look at the tools used to analyze scalar and multivariate functions. The type of tool is what I would typically refer to as topological data structure. An specific example along this line are things like **contour trees, Morse-Smale complexes, Reeb graphs, Reeb space and Mapper**. In some sense all these tools we're going to cover is one way to summarize the structure of real or vector valued functions for future analysis. All these tools can potentially be combined with machine learning techniques, so they complement machine learning techniques. It either gives a new perspective to describe the underlying data, or it can potentially be combined with existing data science techniques. The combination aspect of this actually still has a lot of open questions, I've seen a few efforts to combine with machine learning techniques and there still a large room for development. And that's a quick overview of what's next to come.

17.2 Morse Functions / Morse Theory

Today I'm going to come back to basics. I'm going to lay some of the theoretical foundations to understand scalar function analysis. The idea is that if we have a data point and a scalar or real valued function on it, I would actually like to assume some simplicity to the function. Sometimes it is not enough to assume that the underlying data is smooth. Even if it was continuous, it could be extremely complicated. So a typical thing we will assume even if it might not be true for certain data sets is that the function is simple, and a idea of assuming said simplicity is the Morse function.

And the basics we're going to cover today is typically called Morse theory, and I would consider this as a small topic in mathematics, but it is very important in understanding data in the real world. I'd to emphasis there is two references:

- *An Introduction to Morse Theory*, by Yukio Matsumoto
- *Text Book chapter VI*

As I mentioned before, the idea of Morse function is to put a simplex assumption over the function itself. It turns out that Morse function can be frequently used, as we could assume many function to be Morse, or almost Morse after some permutations. The main idea for **Morse function** is that **study simple, real-valued functions on a manifold.** When I say simple, I mean that it has simple critical points. This is not a definition but more of a goal or aim. When I talk about Morse function, I'm not sitting on a point cloud but a manifold.

$$f : \mathbb{M} \to \mathbb{R}, \quad \mathbb{M} : manifold$$

This will lead to the understanding of what is function. The next step is what is the **general smooth function**. The next step is then from a smooth to a discrete function, then what is known as **piecewise (PC) linear function**. In other words understanding Morse functions enable understanding of more general functions.

17.2.1 Height Function And Level Set

Lets start with the simplest function. **Let $\mathbb{M}$ be a 2 dimensional manifold, which is a torus. let $f : \mathbb{M} \to \mathbb{R}$ be the "height function."** He the torus be vertical on a plane. The function f measures the height of every point on the manifold.

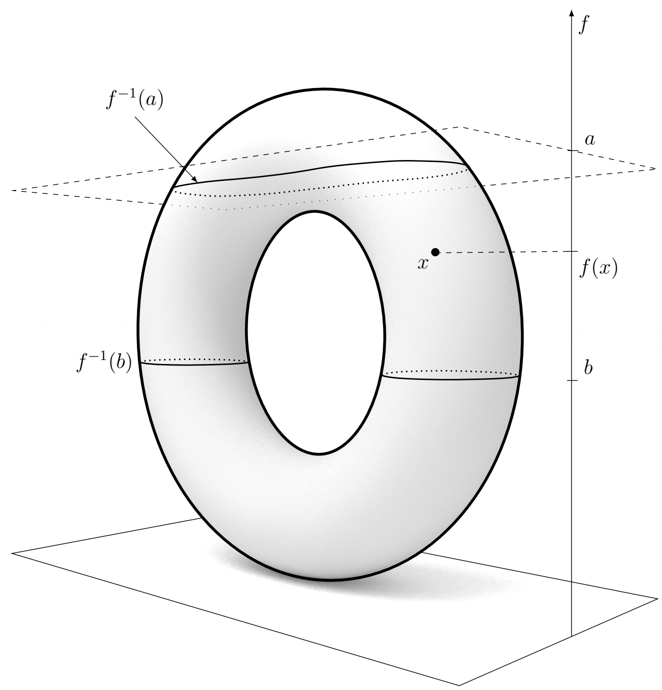

Figure 17.1: The vertical height function on the upright torus

The concept I want to define are **level sets, sublevel sets, superlevel sets and interval-level sets**.

A level set takes any real value in $\mathbb{R}$ and look like inverse.

Definition 17.1. *A **Level set** is the pre-image of* $a \in R$, $f^{-1}(a) = \{x \in \mathbb{M} \mid f(x) = a\}$

Look at all points in the domain that have value a. In the torus, the level set is the slice where points have value a. It essentially a slice plane intersecting manifold.

instead of just looking at intersection, we can look at the points on the manifold with value below it. It consists of all points on the manifold lower than a.

Definition 17.2. *A **sublevel set** consists of all points at height at most a,* $\mathbb{M}_a = f^{-1}(-\infty, a] = \{x \in \mathbb{M} \mid f(x) \leq a\}$.

In the case of torus, the sublevel set looks like a capped torus.

Next we look at a superlevel set. Judging by the naming of this, a superlavel set $\mathbb{M}_a$ is the inverse image of height function above or equal to a.

Definition 17.3. *A **superlevel set** is* $\mathbb{M}^a = f^{-1}[a, \infty) = \{x \in \mathbb{M} \mid f(x) \geq a\}$.

Definition 17.4. *An **interval level set** is* $f^{-1}[a, b] = \{x \in \mathbb{M} \mid a \leq f(x) \leq b\}$

17.2.2 Applications

A sublevel set is useful in persistent homology. When we talk about persistent homology, we need to build a sequence of spaces, one of which is a filtration. An instance of this is a sensor network, when you increase the radius to build a function.

In the setting of Morse, or any scalar functions, we can build a sequence involving sublevel sets or superlevel sets. We essentially start with a small a, and increase it. The sublevel set is the point below a manifold that is below a plane, We can have a raising levelset, resuting in a filtration of sublevelsets.

Filtration involving sublevel sets:

$$f : \mathbb{M} \to \mathbb{R}$$

$$a_1 \leq a_2 \leq a_3 \ldots$$

$\mathbb{M}_{a_1} \to \mathbb{M}_{a_2} \to \mathbb{M}_{a_3}$ is a filtration

$$H(\mathbb{M}_{a_1}) \to H(\mathbb{M}_{a_2}) \to H(\mathbb{M}_{a_3})$$

maps the homology of each filtration

We can thus have a homology group computing the simplest form of persistent homology

17.2.3 Point clouds to function on point could

Data typically exists in the form of point clouds, such as sensor networks. The next setting is having functions on point cloud data.

They are similar - if it is just a point cloud, we are really computing distance function on point cloud to build a filtration based on proximity. Therefore they are equivalent.

If we have a single point in sensor networks, we have a radius a, a disk around the point which can be described as $f^{-1}[0, a]$. In the distance scenario anything less than 0 doesn't make sense. This is equivalent to $[-\infty, a]$. f is the distance function $f : \mathbb{R}^2 \to \mathbb{R}, f(x) = ||X - P||_2, p \in X$

In this scenario, our sublevel set all points that are at most a away from point Increasing the radius is the same as increasing the sublevel set threshold.

17.2.4 Study The Evolution Of Sub-level Set

We go back to a height function on a standing torus. A standing torus has Morse height function, a laying one does not.

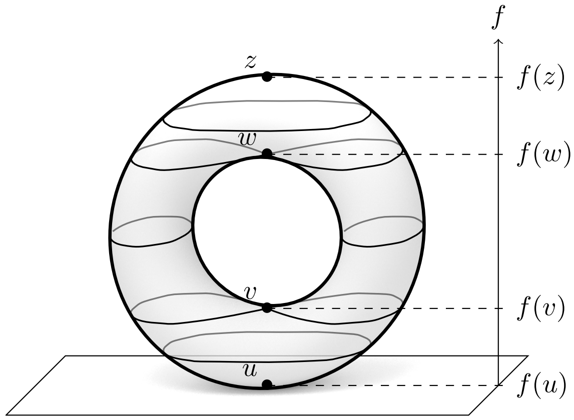

Figure 17.2: The critical points with level sets on the upright torus

The 4 critical points, to be defined later, of the upright torus are: global minimum u, saddle point v and w, and global maximum z.

Review: Two topological space $\mathbb{X}$ & $\mathbb{Y}$ are *homotopy equivalent*, $\mathbb{X} \simeq \mathbb{Y}$, (of the same homotopy type), if there are continuous maps $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{X}$, such that $g \circ f \simeq \mathrm{id}_{\mathbb{X}}$, $f \circ g \simeq \mathrm{id}_{\mathbb{Y}}$. Classical example is donut and cup.

We will now look at the shape of the torus between an at the critical points and their homotopy type. In the graph below, the 1st row depicts the sublevel sets of a torus, while the 2nd row depicts figures that they are homotopy equivalent to.

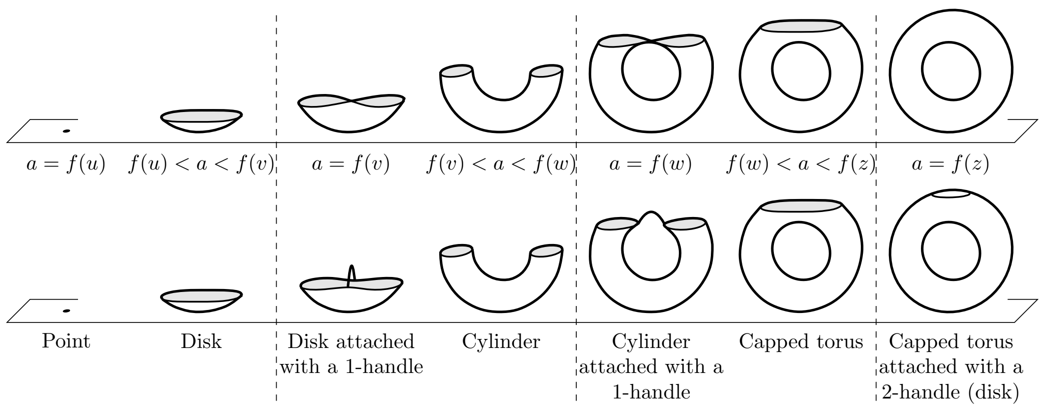

Figure 17.3: The evolution of sub-level set, different homotopy types are divided by dashed lines

Whenever we pass through a critical point, something interesting happens, namely handle attachment. A way to visualize the homotopy equivalence of handles is by broadening the handles. (When going through z, a 2-handle is attached instead of a one handle)

17.2.5 Critical Points

Let $\mathbb{M}$ be a d-dim smooth manifold. (Locally looks like an open ball in $\mathbb{R}^d$.)

Definition 17.5. *A **critical point** of a function $f : \mathbb{M} \to \mathbb{R}$ is a point $x \in \mathbb{R}$ such that the derivative at x is zero.*

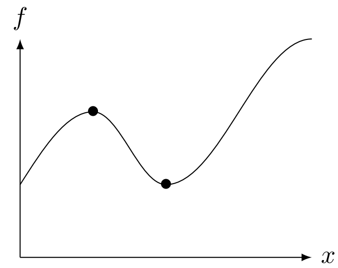

Figure 17.4: The critical point of a function

If we have a local coordinate system $x_1, x_2, ..., x_d$ in a neighborhood of x, then x is critical if and only if all its partial derivatives are zero.

$$\frac{\partial f}{\partial x_1}(x) = \frac{\partial f}{\partial x_2}(x) = ... = \frac{\partial f}{\partial x_d}(x) = 0$$

Definition 17.6. *If x is a critical point, $f(x)$ is a **critical value**, otherwise is **regular value**.*

Values between critical point have homotopy equivalence.