University of Utah
School of Computing

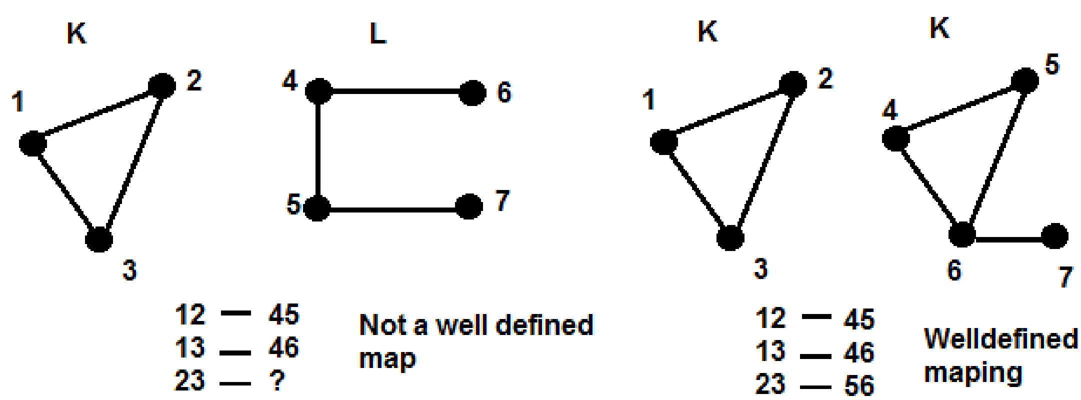
# Lecture 22: Mar 30, 2017

*Lecturer: Prof. Bei Wang* <beiwang@sci.utah.edu> *Scribe: Ram Kashyap S*

## 22.1 Simplicial Map

Let K, L be two finite simplicial complexes over the vertex set $V_K$ and $V_L$

A set map $\phi : V_K \to V_L$ is a simplicial map if $\phi(\sigma) \in L \; \forall \; \sigma \in K$




### 22.1.1 Definition

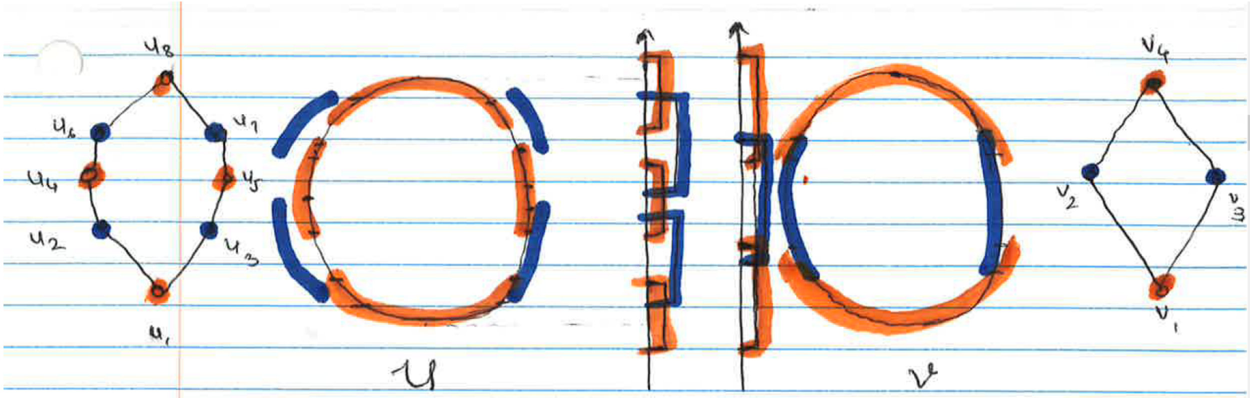
If we have two covers of $\mathbb{X}$

$\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$

$\mathcal{V} = \{V_\beta\}_{\beta \in B}$

A map of covers from $\mathcal{U}$ to $\mathcal{V}$ is a set map $\Gamma : A \to B$
so that $U_\alpha \sqsubseteq V_{\Gamma(\alpha)} \; \forall \alpha \in A$

Given such a map of covers, there is an individual simplicial map
$\Gamma^* : N(\mathcal{U}) \to N(\mathcal{V})$ given on vertices by $\Gamma$



## 22.2 Stability

### 22.2.1 Persistent Diagram:

Multi set of points in the extended plane $\mathbb{R}^2 = (\mathbb{R} \cup \{\pm\infty\})^2$

contains finite number of points off the diagonal and infinite points on the diagonal

### 22.2.2 $L_\infty$ norm:

For two points $\mathcal{X} = (x_1, x_2)$ , $\mathcal{Y} = (y_1, y_2)$ $L_\infty$ norm is defined as

$\|x - y\|_\infty$ = max$\{|x_1 - y_1|, |x_2 - y_2|\}$

### 22.2.3 Bottleneck Distance:

Let X, Y be two persistent diagrams with $\eta : X \to Y$ as a bijection then bottle neck distance

$W_\infty(X, Y) = inf \;\; sup\|X - \eta(X)\|_\infty$

where the inf is over all possible bijections and
the sup is over all $x \in X$
Bottle Neck distance is a metric. It satisfies the following properties:

- $W_\infty(X, Y) = 0 \;\; iff \;\; X = Y$
- $W_\infty(X, Y) = W_\infty(Y, X)$
- $W_\infty(X, Z) = W_\infty(X, Y) + W_\infty(Y, Z)$

### 22.2.4 Stability of a tame function:

**Theorem 22.1.** *Let $\mathbb{X}$ be a triangulable topological space and $f, g :\to R$ be two tame functions for each dimension p,*

$W_\infty(Dgm_p(f), Dgm_p(g)) \leq \|f - g\|_\infty$

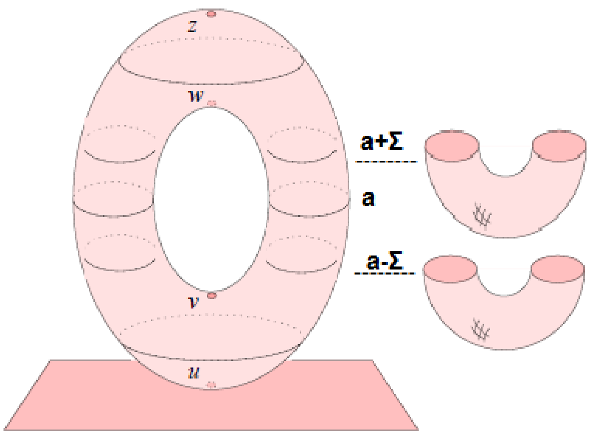
### 22.2.5 Definition: Tame function

A function $f: \mathbb{X} \to \mathbb{R}$ is tame if it has a finite number of homological critical values and the homological groups of all sub level sets have finite rank

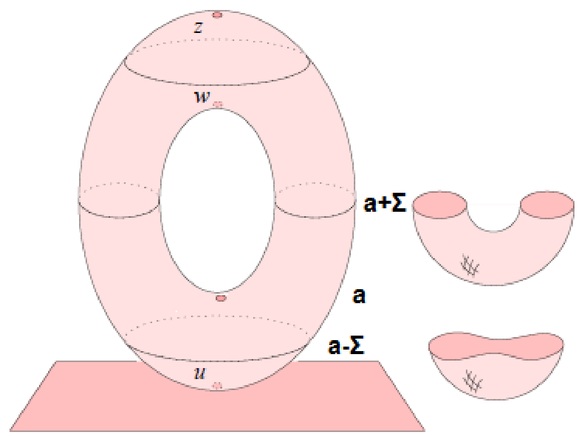
### 22.2.6 Definition: Homological Critical Value

A point $a \in \mathbb{R}$ is a homological critical value if there is no $\epsilon > 0$ for which $f_P^{a-\epsilon, a+\epsilon}$ is an isomorphism for each P

$f_P^{a,b}: H_P(\mathbb{X}_a) \to H_P(\mathbb{X})$
$\mathbb{X}_{(a)} = f^{-1}(-\infty, a]$




a is not a h.c.v



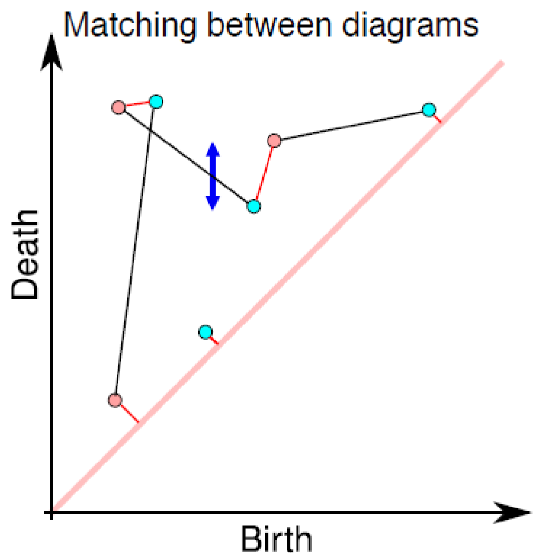

a is a h.c.v

### 22.2.7 Definition: Wasserstein distance

Degree - q wasserstein distance between two persistence diagrams is given as

$$W_q(X, Y) = \left( \inf_{\eta:\ X \to Y} \sum_{x \in X} \|x - \eta(x)\|_\infty^q \right)^{\frac{1}{q}}$$

”Transportation problem” minimizes the cost of moving or transporting all $*$ points to corresponding ”.” points. Correspondence is given by bijection $\eta$. Minimize over all possible $\eta$




### 22.2.8 Stability bound:

$W_\infty(Dgm_p(f), Dgm_p(g)) \leq \|f - g\|_\infty^{1-\frac{k}{q}} for\ q \geq k > j$

C and K are constants. $f, g : \mathbb{X} \to \mathbb{R}$ are tame, Lipschitz functions on metric spaces whose triangulations grow polynomially with constant exponent g.

$\exists$ constants c, j such that $N(r) \leq \frac{c}{r^j}$ where k: simplicial complex N(r): number of simplexes with maximum diameter at most r