CS 6170: Computational Topology, Spring 2019
Lecture 09

Topological Data Analysis for Data Scientists

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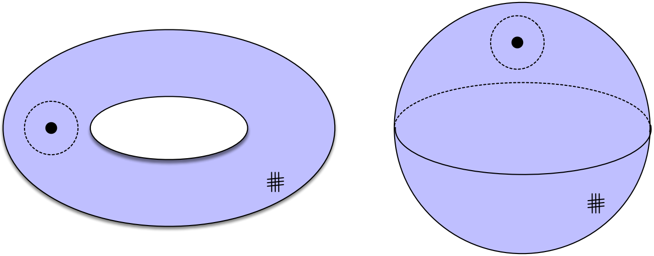
2-dimensional Manifold

Book Chapter A.II

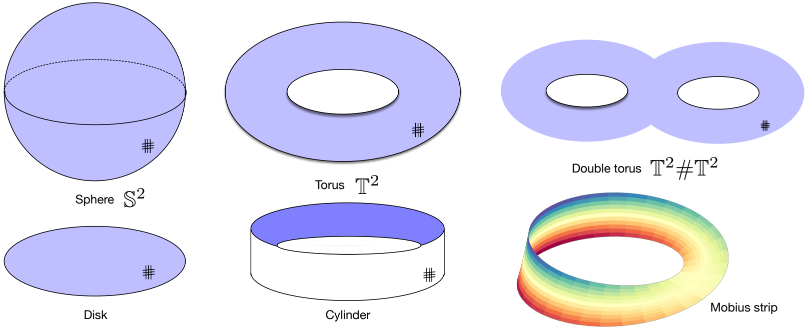
2-manifold without boundary

- A *2-manifold without boundary* is a topological space $\mathbb{M}$ whose points all lie in open disks.
- Intuitively, this means that $\mathbb{M}$ locally looks like a plane.
- We get a *2-manifold with boundary* by removing open disks from a 2-manifolds without boundary.



Examples of 2-manifolds

- Top: 2-manifold without boundary
- Bottom: 2-manifold with boundary
- Möbius strip: non-orientable manifold; 2 sides locally, 1 side globally.
- Möbius strip: an ant will travel all surface area
- Möbius strip: its boundary is a single circle
- Quiz: what happens if you cut Mobius strip along its center line?



https://commons.wikimedia.org/wiki/File:M%C3%B6bius_strip.png

Orientability

- If all closed curves in a 2-manifold are orientation-preserving, then the 2-manifold is *orientable*.
- Creating compact 2-manifolds using *polygonal schema*.
- $\mathbb{M}$ is compact if for every covering of $\mathbb{M}$ by open sets, called an open cover, we can find a finite number of the sets that cover $\mathbb{M}$.
- A subset of Euclidean space is *compact* if it is closed and bounded (i.e., contained in a ball of finite radius).

Classification

Theorem (Classification theorem for compact 2-manifolds)

The two infinite families $\mathbb{S}^2$, $\mathbb{T}^2$, $\mathbb{T}^2 \# \mathbb{T}^2, \cdots$, *and* $\mathbb{P}^2$, $\mathbb{P}^2 \# \mathbb{P}^2, \cdots$, *exhaust the compact* 2*-manifolds without boundary.*

(Edelsbrunner and Harer, 2010, Page 29)

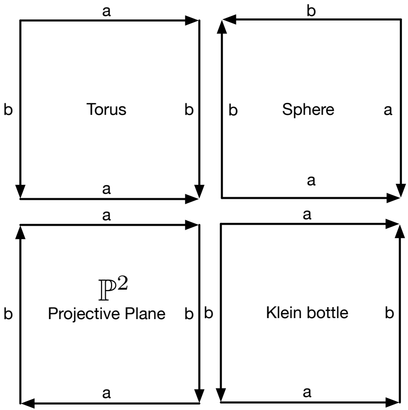
Classification

Any connected closed surface is homeomorphic to some member of one of these three families:

- The sphere
- The connected sum of g tori, for $g \geq 1$
- The connected sum of k real projective planes, for $k \geq 1$.

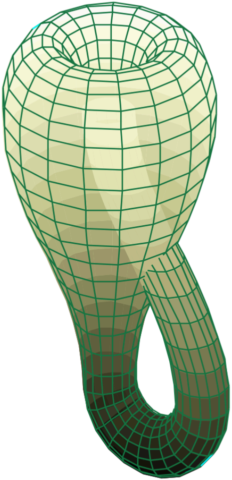
https://en.wikipedia.org/wiki/Surface_(topology)#Classification_of_closed_surfaces

Polygonal schema



- Projective plane: glue a disk to a Möbius strip
- Klein bottle: glue 2 Möbius strips together

Klein bottle: non-orientable surface



https://en.wikipedia.org/wiki/Klein_bottle

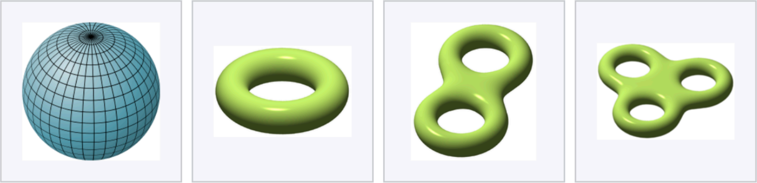
Later: show up in data analysis of natural image patches Carlsson et al. (2008).

Betti numbers β_i of 2-manifolds

	β_0	β_1	β_2
circle	1	1	0
sphere	1	0	1
torus	1	2	1
projective plane	1	0	0
Klein bottle	1	1	0
2-hole torus	1	4	1
g-hole torus	1	2g	1

Genus

- The *genus* of a connected, orientable surface is an integer representing the maximum number of cuttings along non-intersecting closed simple curves without disconnecting the resulting manifold.



genus 0 genus 1 genus 2 genus 3

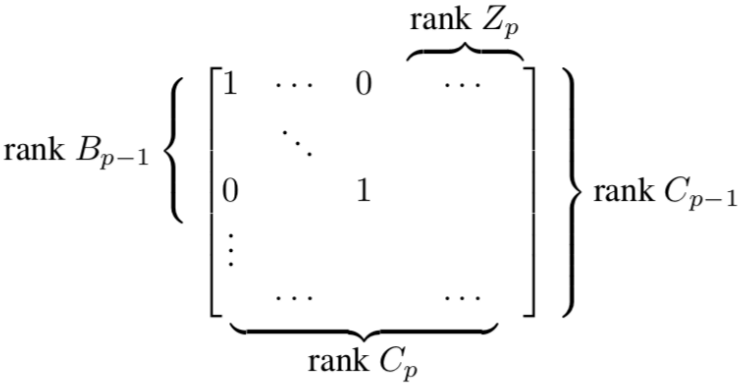
`https://en.wikipedia.org/wiki/Genus_(mathematics)`, also for further reading

Computing Homology

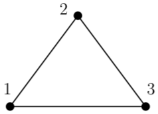
Book Chapter B.IV.

Reduction of a boundary matrix

- Let ∂_p be the p-th boundary matrix
- After reduction (row and column operation), ∂_p turns out to be a matrix N_p in Smith Normal Form
- `https://en.wikipedia.org/wiki/Smith_normal_form`
- $\beta_p = \text{rank } Z_p - \text{rank } B_p$



Example: a triangulation of a circle



$$\partial_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 12 & 23 & 31 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right] \end{array}$$

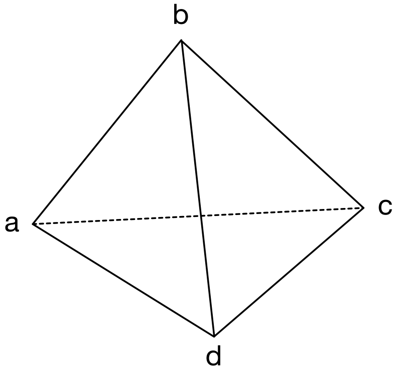
$$\Rightarrow \begin{array}{c} \\ 1 \\ 2(+1) \\ 3(+2) \end{array} \begin{array}{c} \begin{array}{ccc} 12 & 23 & 31(+23) \end{array} \\ \left[\begin{array}{ccc} \boxed{1} & 0 & 1 \\ \not{1} & \boxed{1} & \not{1} \\ 0 & \not{1} & \not{1} \end{array}\right] \end{array}$$

$$\mathcal{N}_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 12 & 23 & 31 \end{array} \\ \left[\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array}$$

- $\partial_0 = N_0$ is a 1×3 matrix with all 0 entries.
- $\text{rank } C_0 = \text{rank } Z_0 = 3$; $\text{rank } Z_1 = 1$, $\text{rank } B_0 = 2$
- $\beta_0 = \text{rank } Z_0 - \text{rank } B_0 = 3 - 2 = 1$
- $\beta_1 = \text{rank } Z_1 - \text{rank } B_1 = 1 - 0 = 1$

Take home exercise

The following simplicial complex contains 4 vertices, 6 edges, 3 triangles. Compute its Betti numbers: $\beta_0 = 1$, $\beta_1 = 0$, $\beta_2 = 1$.



References I

Carlsson, G., Ishkhanov, T., De Silva, V., and Zomorodian, A. (2008). On the local behavior of spaces of natural images. *International journal of computer vision*, 76(1):1–12.

Edelsbrunner, H. and Harer, J. (2010). *Computational Topology: An Introduction*. American Mathematical Society, Providence, RI, USA.