Advanced Data Visualization

CS 6965

Spring 2018

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University of Utah

Lecture 16

About…

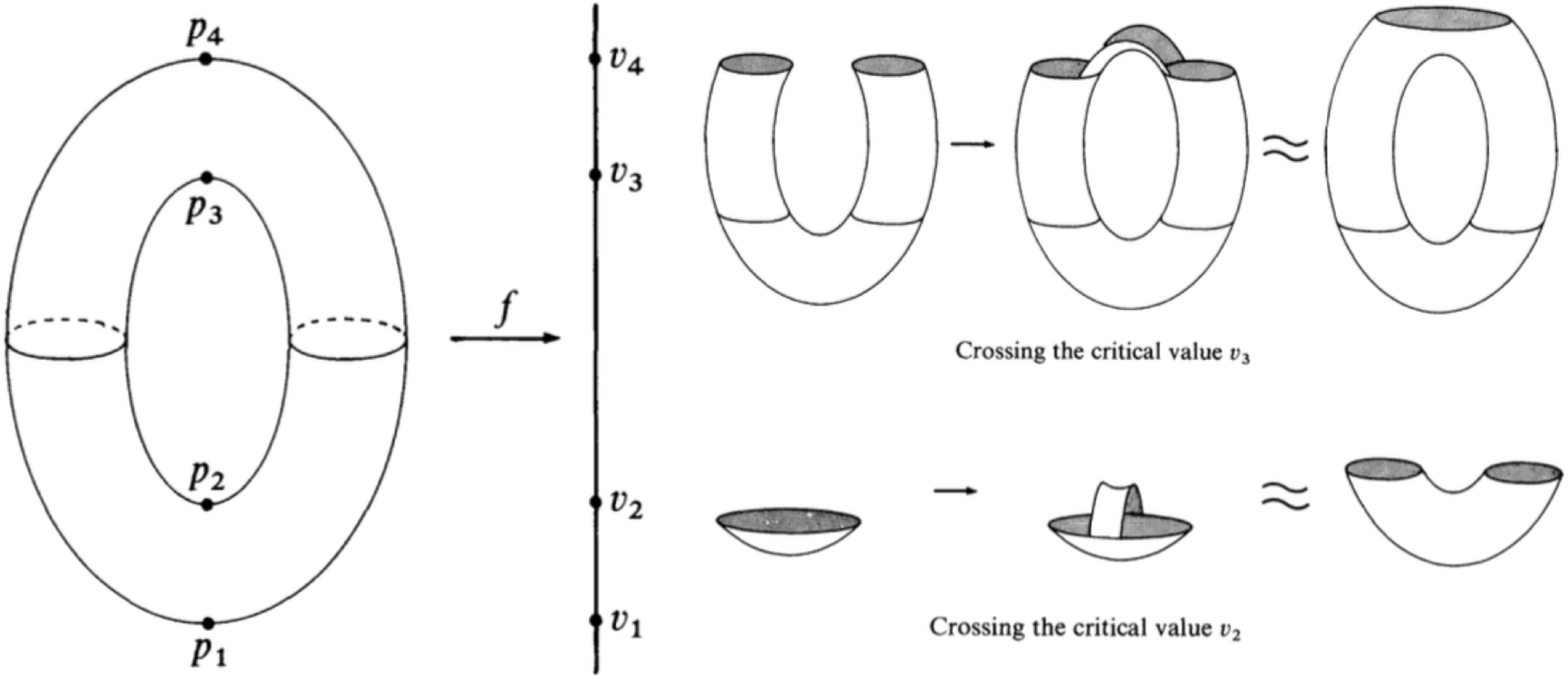
- This lecture can be mathematical…the goal is to give a brief introduction to Discrete Morse Theory and its applications

An Introduction to Discrete Morse Theory

TOPO

Classic Morse Theory (CMT)

CMT Example

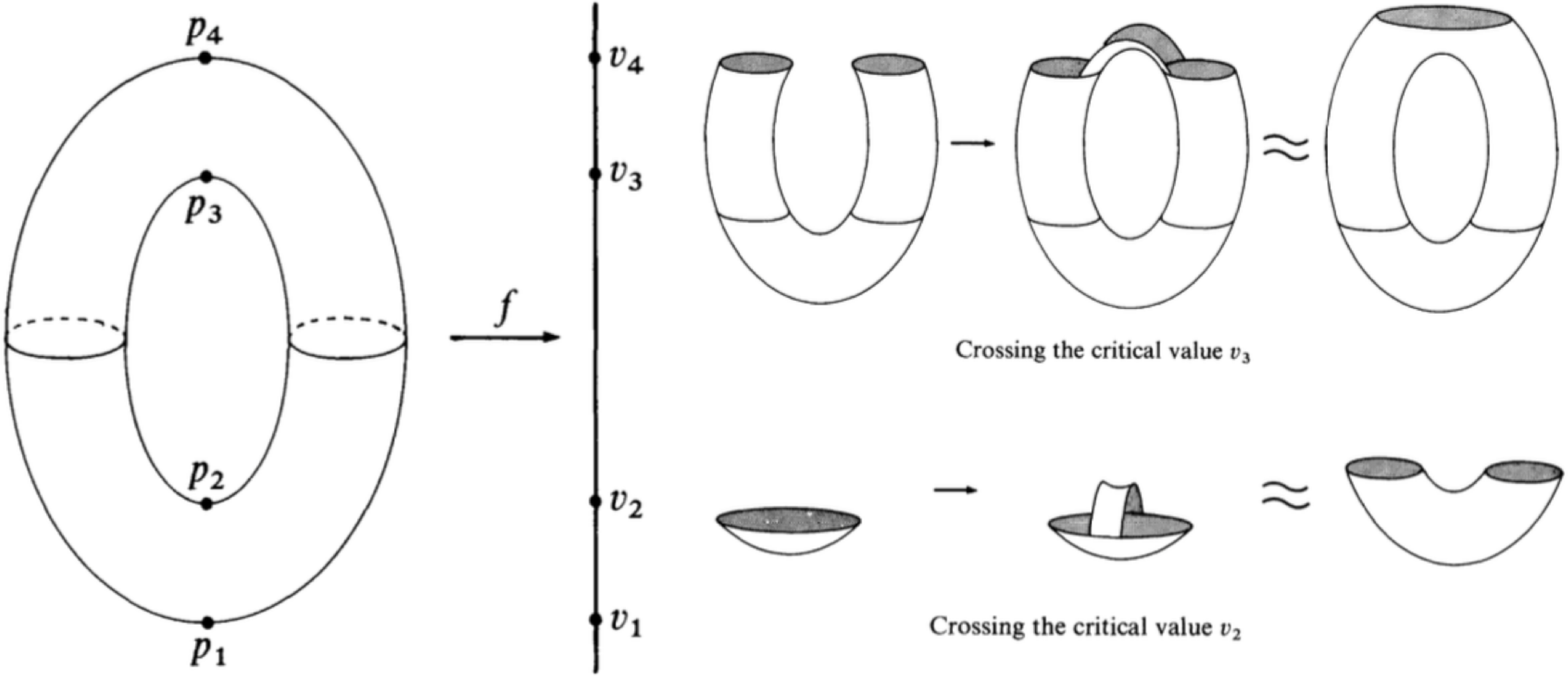


[GoreskyMacPherson1988]

Definitions

CMT studies the topological change of $\mathbb{X}_a$ as a varies.

- $\mathbb{X}$: a compact, smooth d-manifold
- $f : \mathbb{X} \to \mathbb{R}$: differentiable
- *sublevel set*: $\mathbb{X}_a = f^{-1}(-\infty, a]$
- A point $x \in \mathbb{X}$ is *critical* if the derivative at x equals zero
- $\lambda(x)$: the *Morse index* of a non-degenerate critical point x is the number of negative eigenvalues in the Hessian matrix
- Next page: p_1, p_2, p_3, p_4, index 0, 1, 1, and 2
- f is a *Morse function* if all critical points are non-degenerate and its values at the critical points are distinct



[GoreskyMacPherson1988]

Fundamental Results of CMT

Theorem (CMT-A)

Let $f : \mathbb{X} \to \mathbb{R}$ *be a differentiable function on a compact smooth manifold* $\mathbb{X}$.

Let $a < b$ *be real values such that* $f^{-1}[a, b]$ *is compact and contains no critical points of* f.

Then $\mathbb{X}_a$ *is diffeomorphic to* $\mathbb{X}_b$.

Fundamental Results of CMT

Theorem (CMT-B)

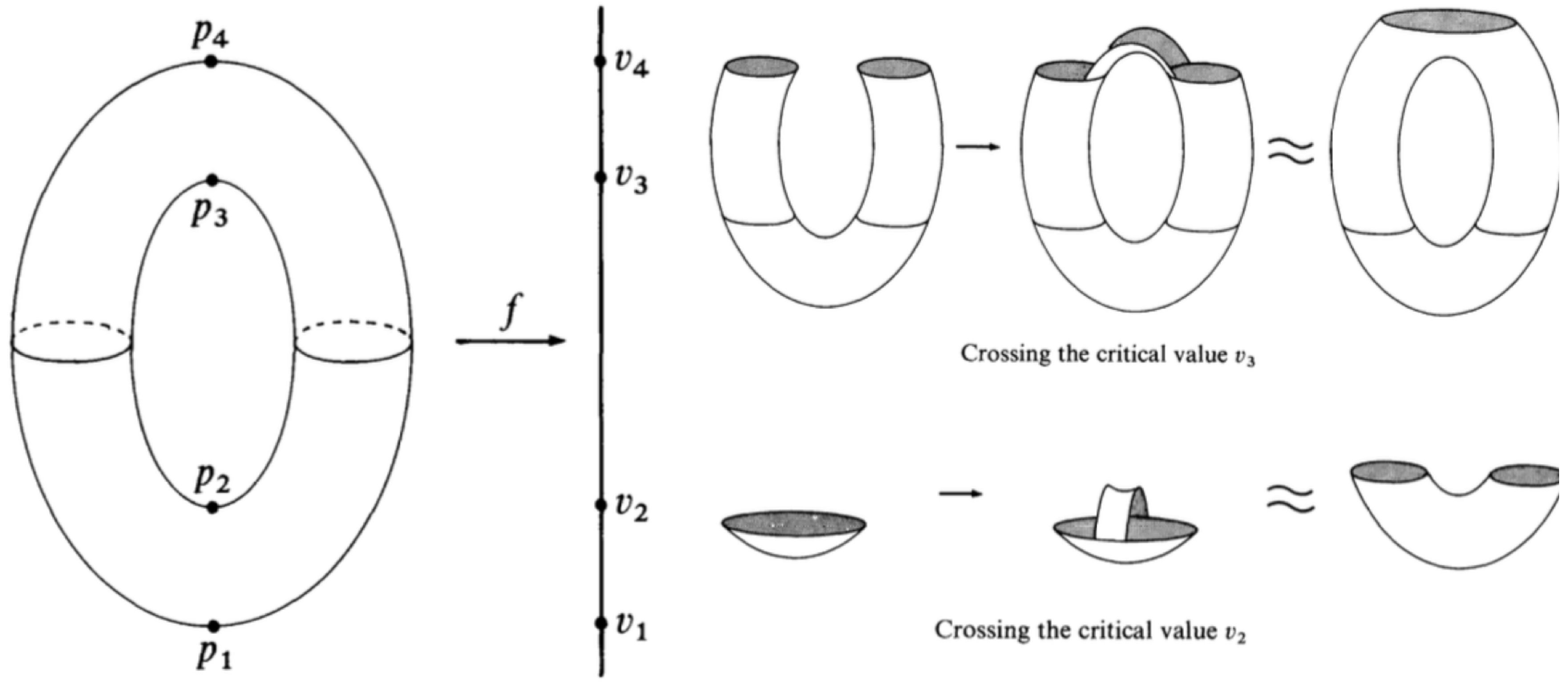
Let f be a Morse function on $\mathbb{X}$.

Consider two regular values $a < b$ such that $f^{-1}[a, b]$ is compact but contains one critical point u of f, with index λ.

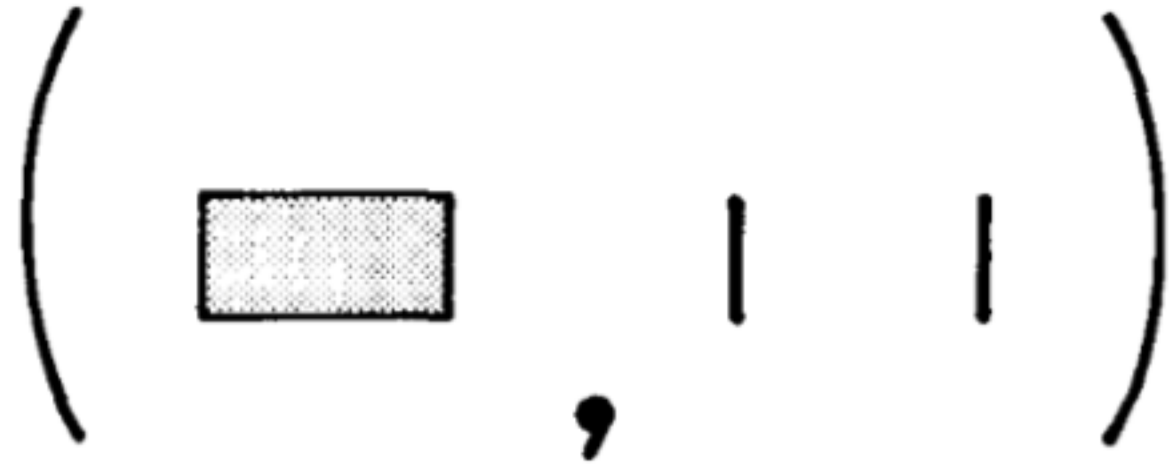
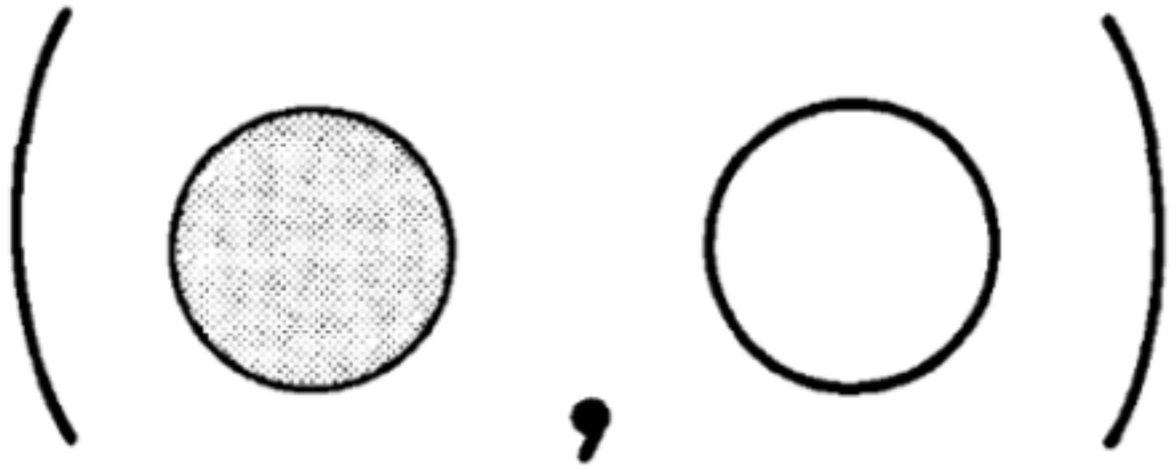
Then $\mathbb{X}_b$ is homotopy equivalent (diffeomorphic) to the space $\mathbb{X}_a \cup_B A$, that is, by attaching A along B.

The pair of spaces $(A, B) = (D^\lambda \times D^{d-\lambda}, (\partial D^\lambda) \times D^{d-\lambda})$ is the Morse data, where d is the dimension of $\mathbb{X}$ and λ is the Morse index of u, D^k denotes the closed k-dimensional disk and ∂D^k is its boundary.

$$(A, B) = (D^\lambda \times D^{d-\lambda}, (\partial D^\lambda) \times D^{d-\lambda})$$



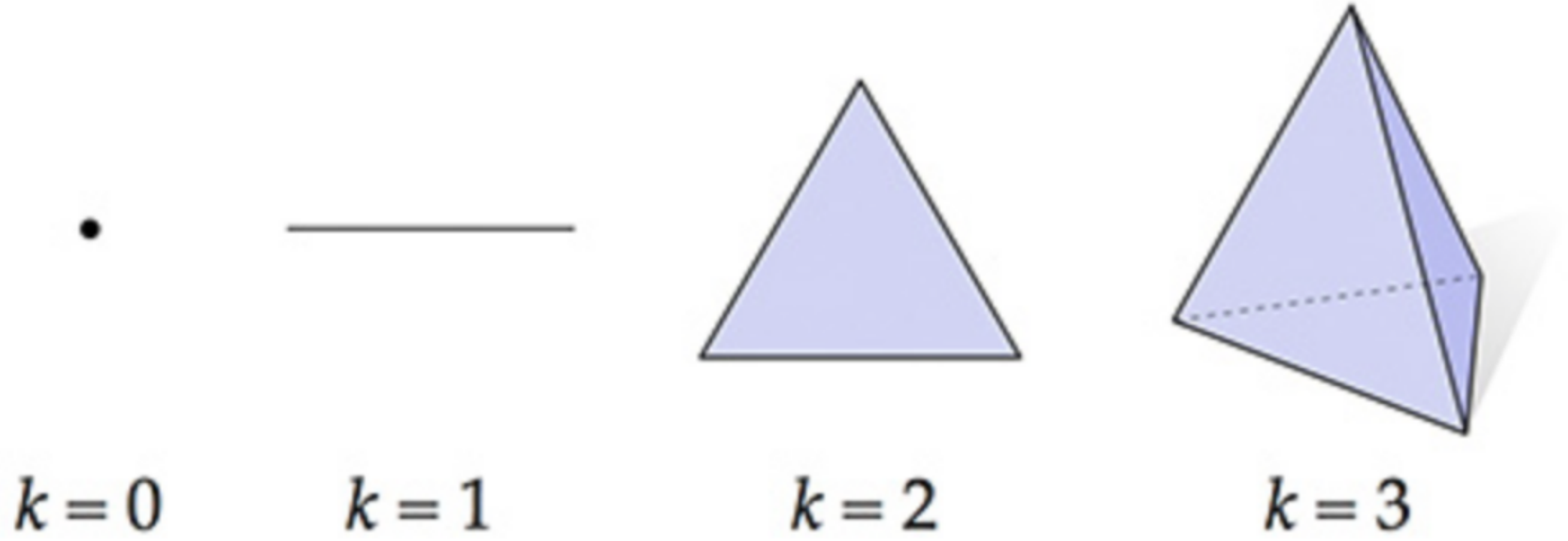
[GoreskyMacPherson1988]

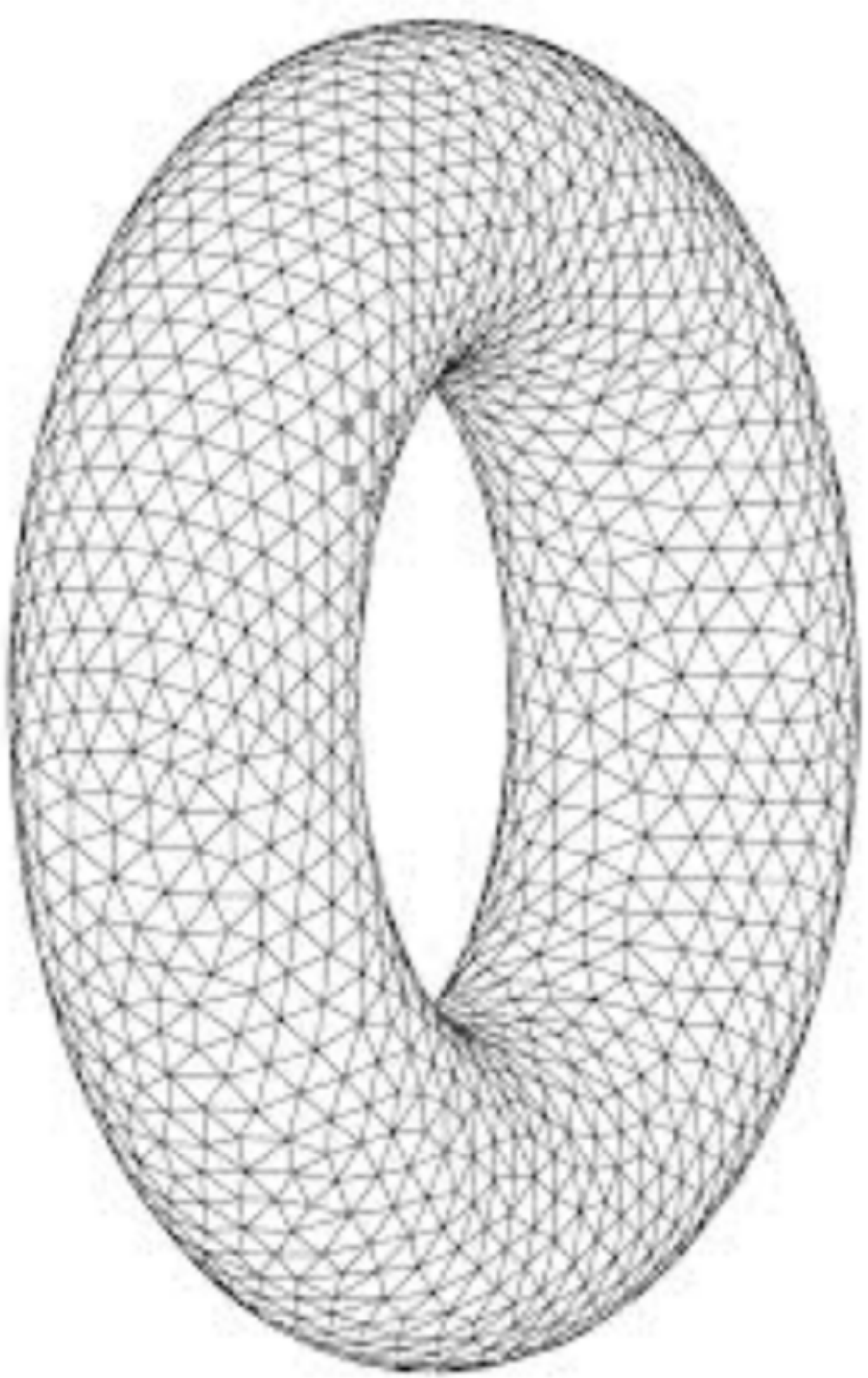
Critical point	Morse data (A, B)
p_1	 $=(D^0 \times D^2, \partial D^0 \times D^2)$
p_2 or p_3	 $=(D^1 \times D^1, \partial D^1 \times D^1)$
p_4	 $=(D^2 \times D^0, \partial D^2 \times D^0)$

[GoreskyMacPherson1988]

Discrete Morse Theory (DMT)

Data as a simplicial complex





DMT Basic Setup

DMT is the combinatorial version of CMT.

- K: a finite simplicial complex
- $\alpha^{(p)} \in K$: a simplex of dimension p.
- $\alpha < \beta$: α is a face of simplex β.
- $U(\alpha) = \{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\}$
- $L(\alpha) = \{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\}$
- $U(\alpha)$ contains the higher-dimensional cofaces of α with lower (or equal) function values
- $L(\alpha)$ contains the lower-dimensional faces of α with higher (or equal) function values.
- Let $|U(\alpha)|$ and $|L(\alpha)|$ be their sizes.

Discrete Morse Function

Definition

A function $f : K \to \mathbb{R}$ is a *discrete Morse function* if for every $\alpha^{(p)} \in K$,

- (i) $|U(\alpha)| \leq 1$ and
- (ii) $|L(\alpha)| \leq 1$.

Definition

A simplex $\alpha^{(p)}$ is *critical* if (i) $|U(\alpha)| = 0$ and (ii) $|L(\alpha)| = 0$. A *critical value* of f is its value at a critical simplex.

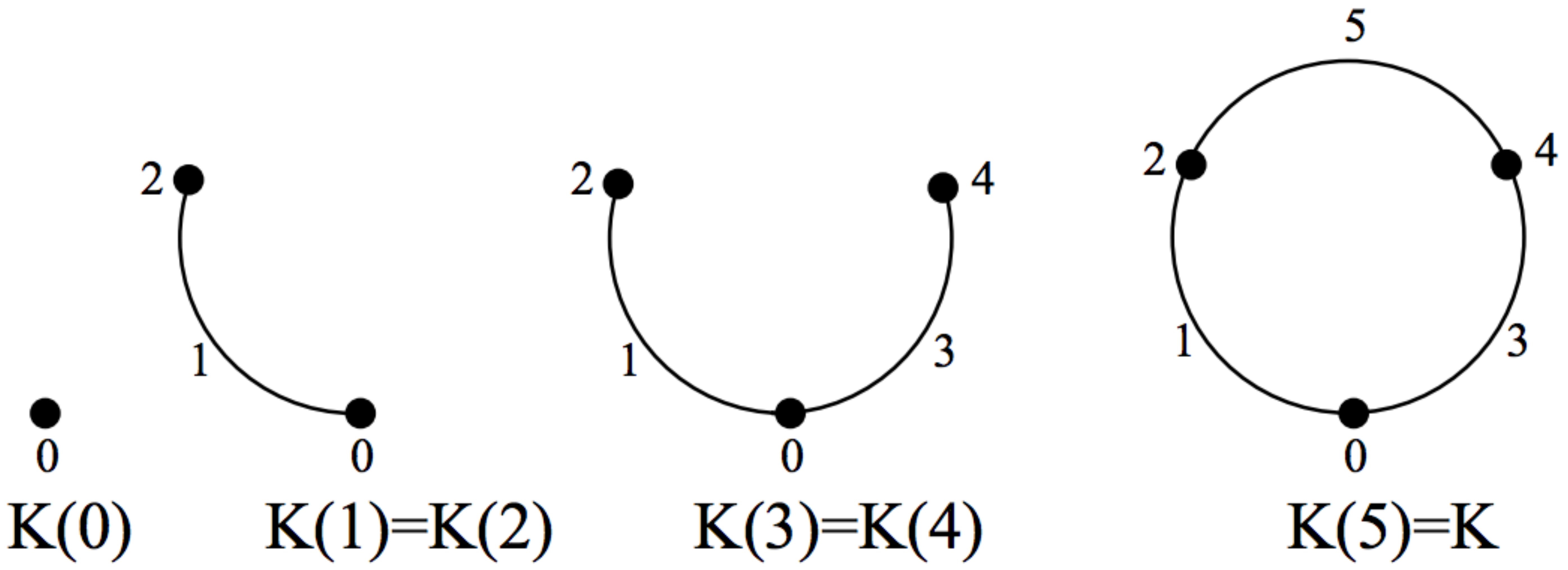
Discrete Morse Function

Definition

A simplex $\alpha^{(p)}$ is *noncritical* if either of the following conditions holds: (i) $|U(\alpha)| = 1$; (ii) $|L(\alpha)| = 1$; as noted above these conditions can not both be true ([Forman1998], Lemma 2.5).

Level Subcomplex

- Given $c \in \mathbb{R}$, we have the *level subcomplex* $K_c = \cup_{f(\alpha) \leq c} \cup_{\beta \leq \alpha} \beta$.
- K_c contains all simplicies α of K such that $f(\alpha) \leq c$ along with all of their faces.



The level subcomplexes of the discrete Morse function shown in Figure 2.2(ii)

Fundamental Results of DMT

Theorem (DMT-A)

Suppose the interval $[a, b]$ *contains no critical value of* f*. Then* K_b *is homotopy equivalent to* K_a*. In fact,* K_a *is a deformation retract of* K_b *and moreover,* K_b *simplicially collapses onto* K_a*.*

Fundamental Results of DMT

Theorem (DMT-A)

Suppose the interval $[a, b]$ *contains no critical value of* f. *Then* K_b *is homotopy equivalent to* K_a. *In fact,* K_a *is a deformation retract of* K_b *and moreover,* K_b *simplicially collapses onto* K_a.

Fundamental Results of DMT

Theorem (DMT-B)

Suppose $\sigma^{(p)}$ *is a critical simplex with* $f(\sigma) \in (a, b]$, *and there are no other critical simplices with values in* $(a, b]$. *Then* $K(b)$ *is homotopy equivalent to attaching a* p*-cell* $e^{(p)}$ *along its entire boundary; that is,* $K_b = K_a \cup_{\dot{e}^{(p)}} e^{(p)}$.

Discrete Gradient Vector Fields

- Given a discrete Morse function $f : K \to \mathbb{R}$ we may associate a discrete gradient vector field.
- Since any noncritical simplex $\alpha^{(p)}$ has at most one of the sets $U(\alpha)$ and $L(\alpha)$ nonempty, there is a unique face $\nu^{(p-1)} < \alpha$ with $f(\nu) \geq f(\alpha)$ or a unique coface $\beta^{(p+1)} > \alpha$ with $f(\beta) \leq f(\alpha)$.
- Denote by V the collection of all such pairs $\{\sigma < \tau\}$.
- Then every simplex in K is in at most one pair in V and the simplices not in any pair are precisely the critical cells of the function f.
- We call V the *gradient vector field associated to* f.
- We visualize V by drawing an arrow from α to β for every pair $\{\alpha < \beta\} \in V$.

Discrete Gradient Vector Fields

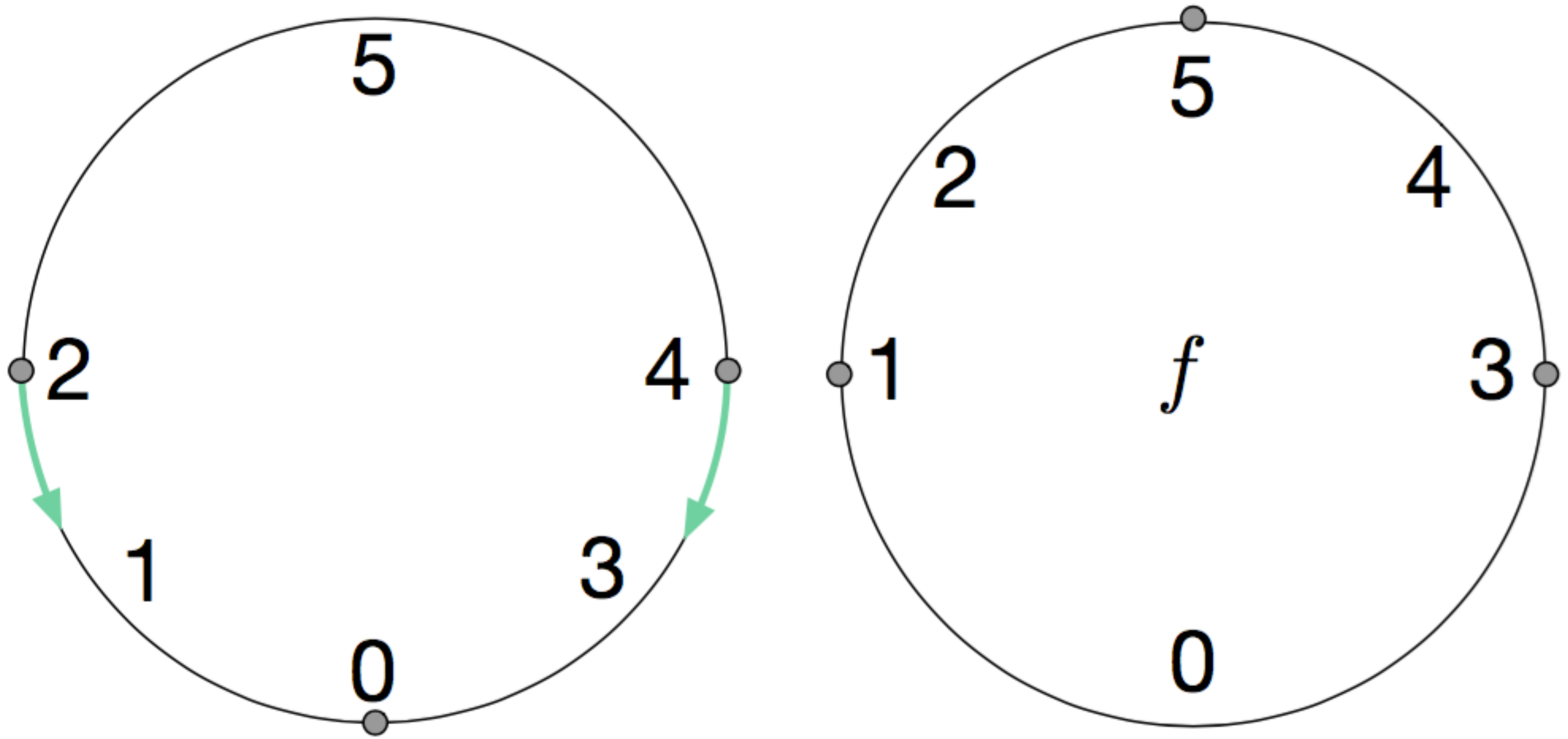
- Data reduction: collapsing the pairs in V using the arrows.
- By a V-*path*, we mean a sequence

$$\alpha_0^{(p)} < \beta_0^{(p+1)} > \alpha_1^{(p)} < \beta_1^{(p+1)} > \cdots < \beta_r^{(p+1)} > \alpha_{r+1}^{(p)}$$

where each $\{\alpha_i < \beta_i\}$ is a pair in V. Such a path is *nontrivial* if $r > 0$ and *closed* if $\alpha_{r+1} = \alpha_0$.

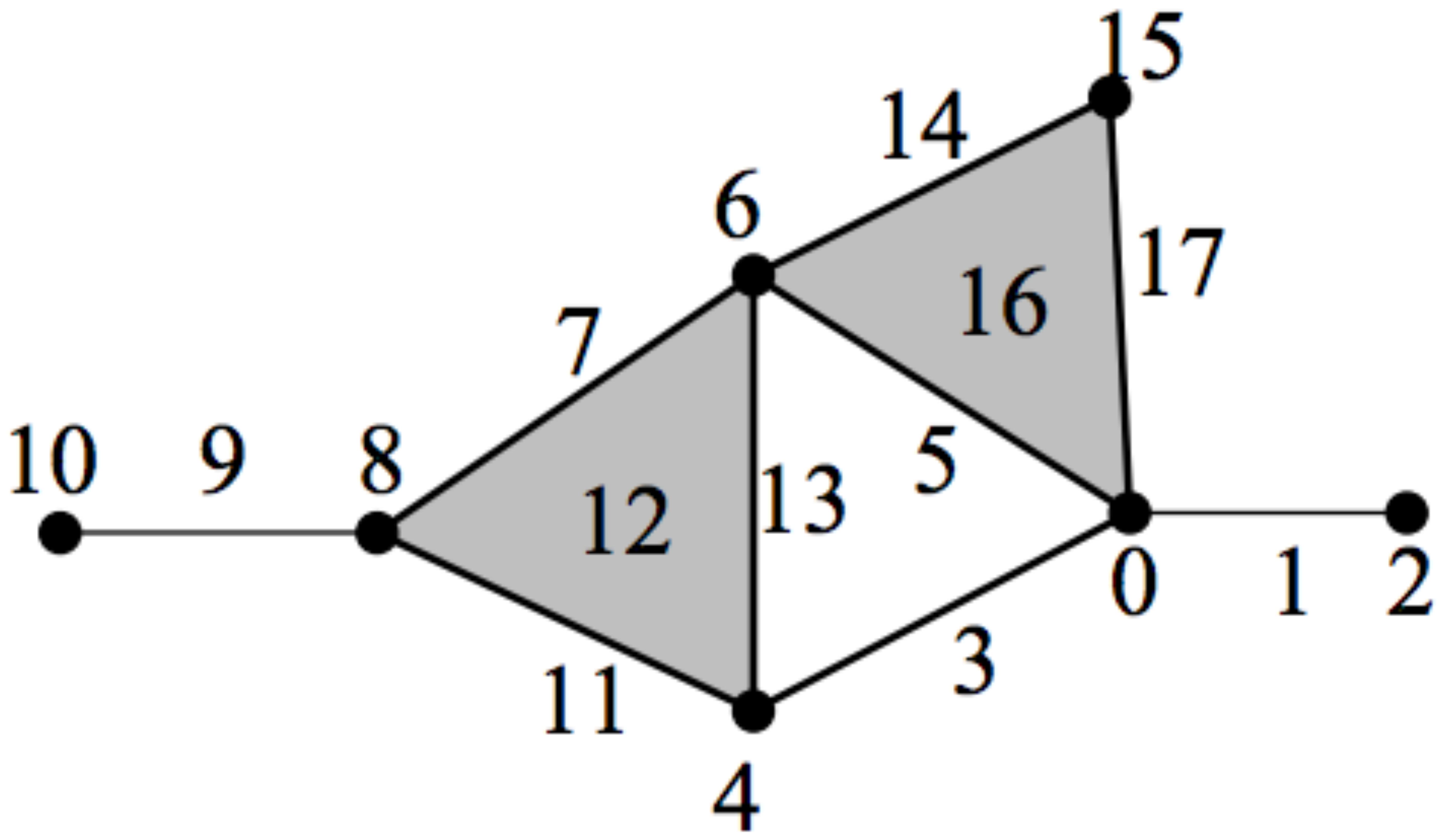
Theorem ([Forman1998])

If V is a gradient vector field associated to a discrete Morse function f on K, then V has no nontrivial closed V-paths.

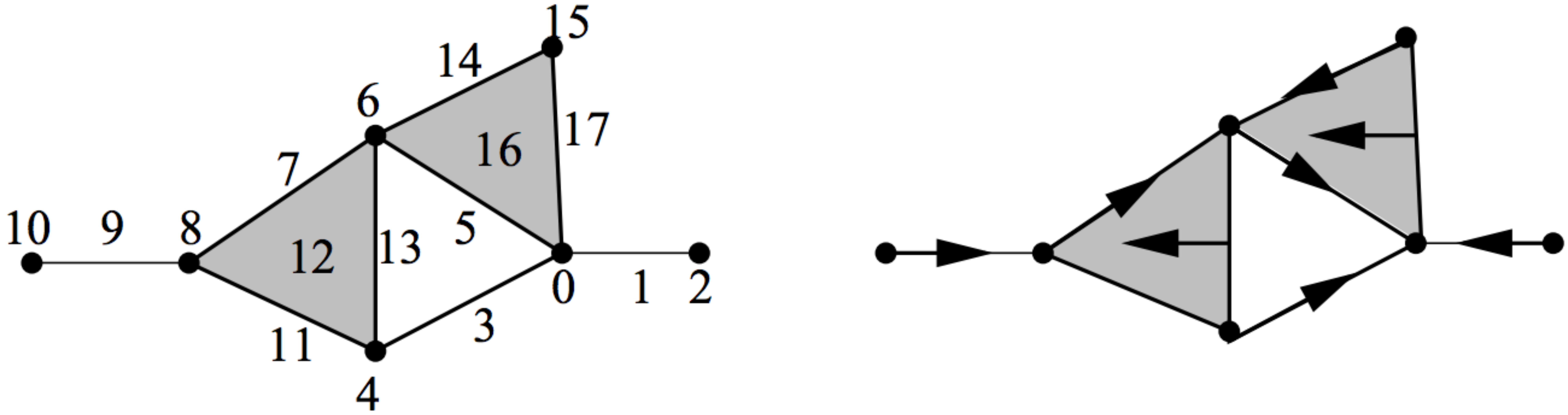


Forman 2002

Draw the DVF

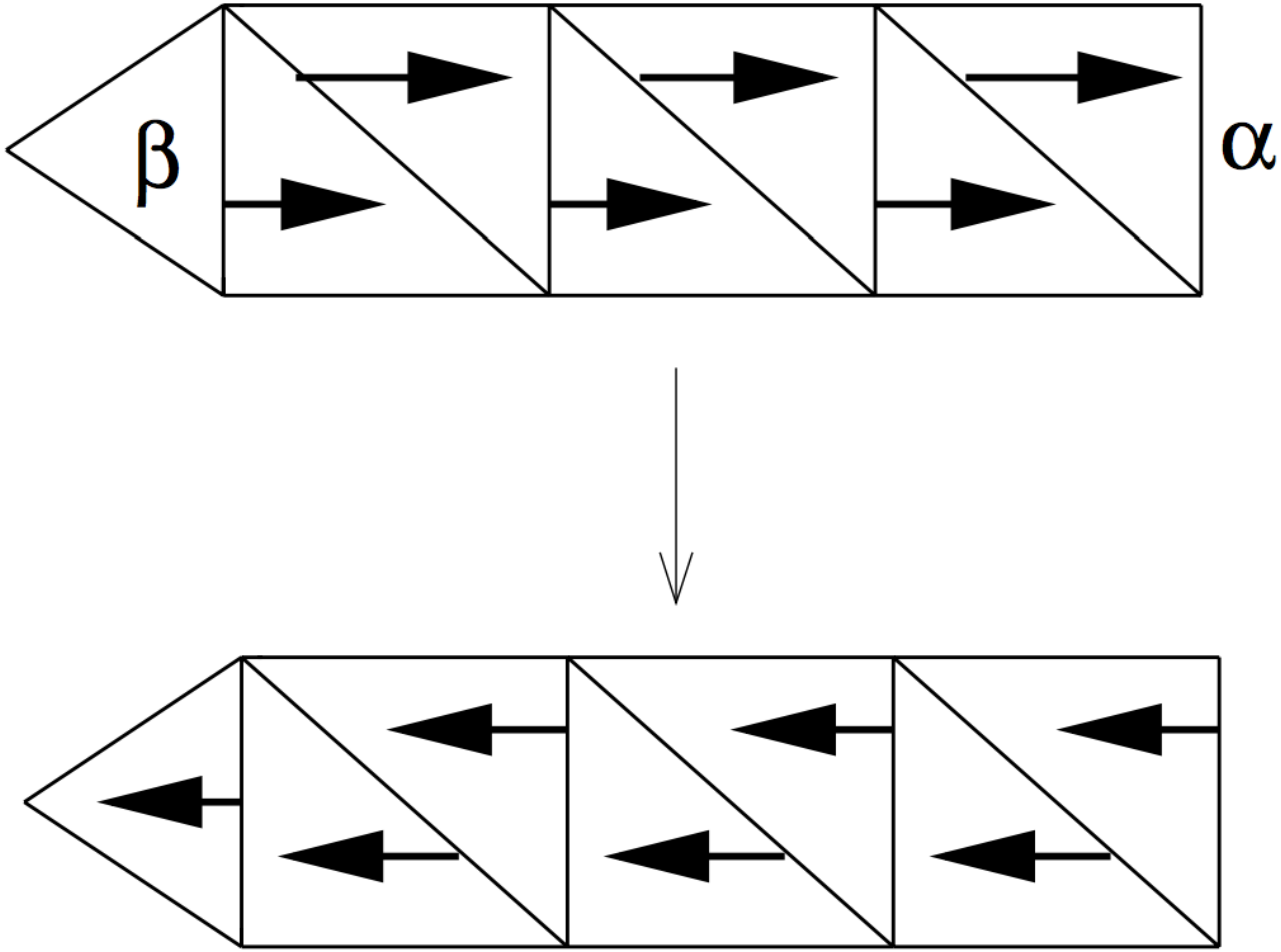


Forman 2002



Forman 2002

DMT
Examples



Forman 2002

Cancelling critical points.

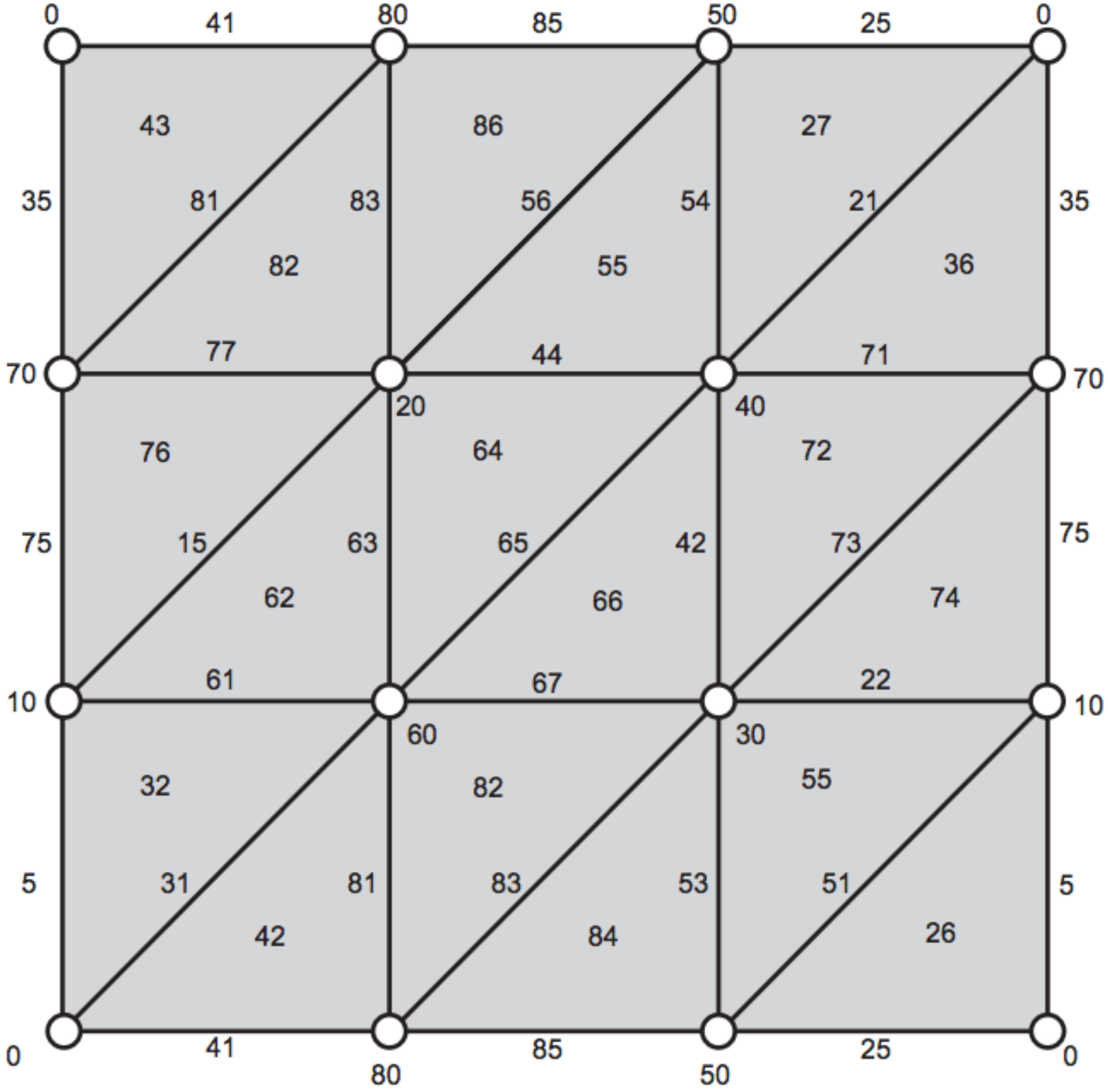


FIGURE 1. A discrete Morse function on the torus.

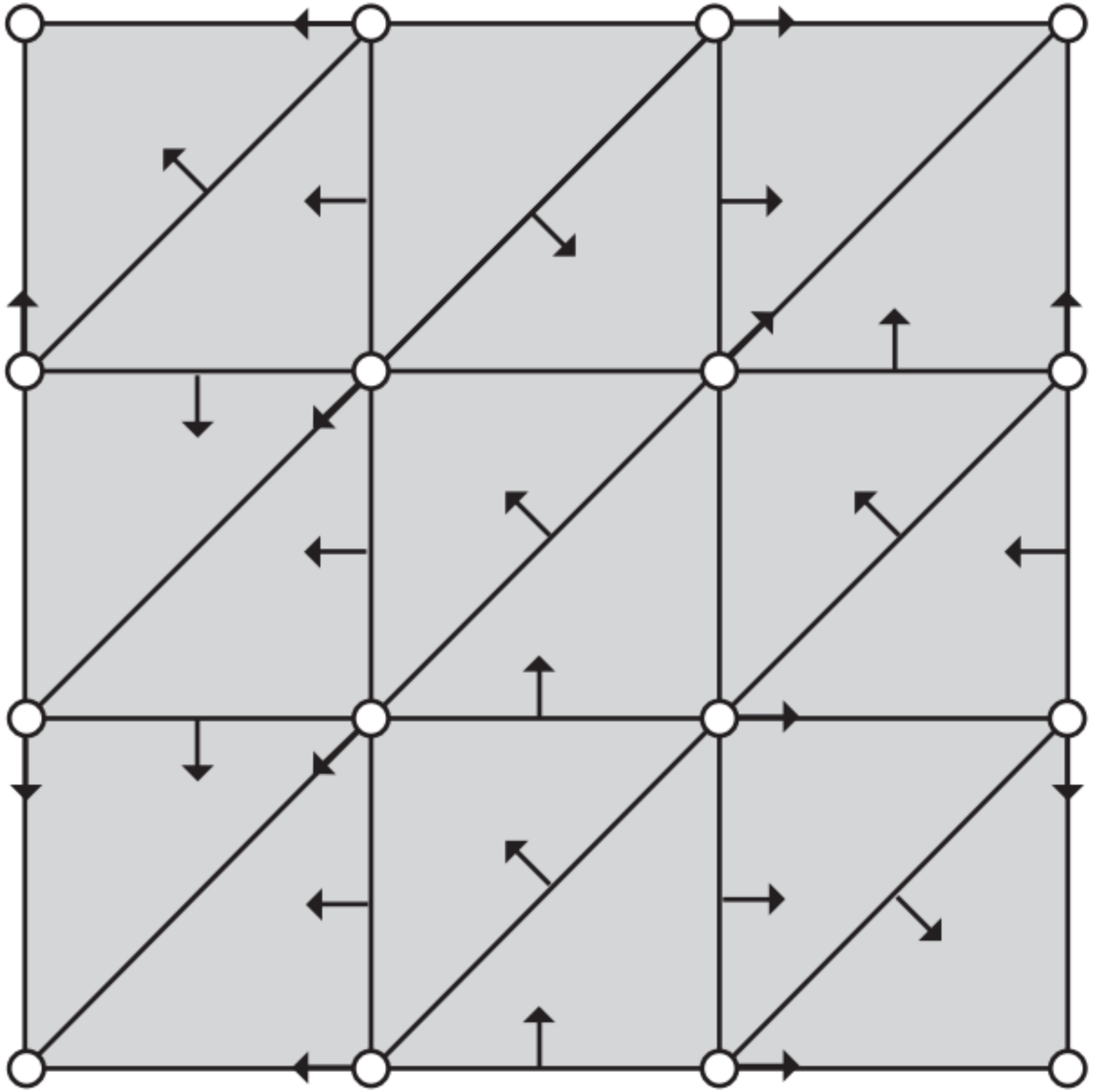
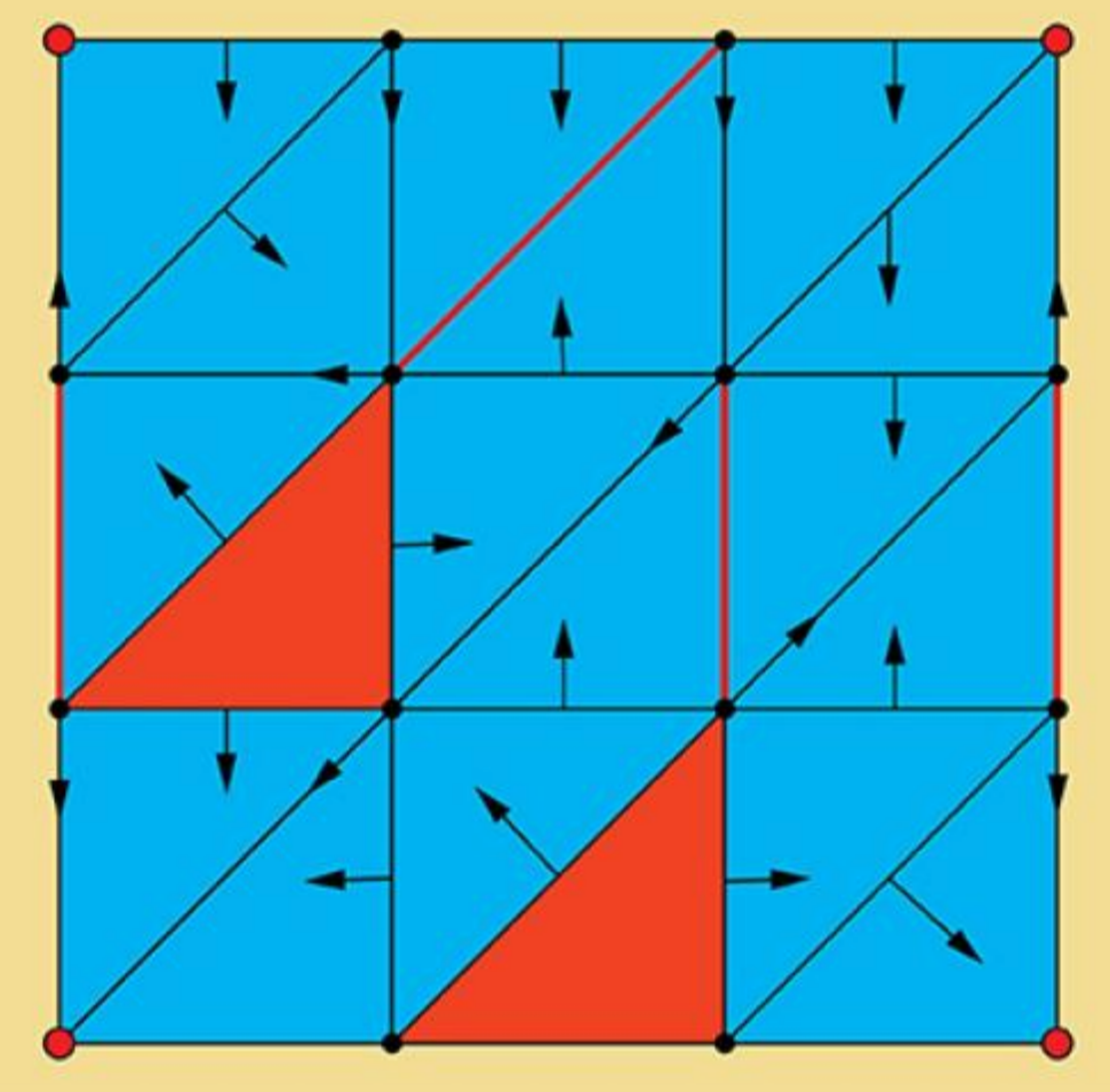
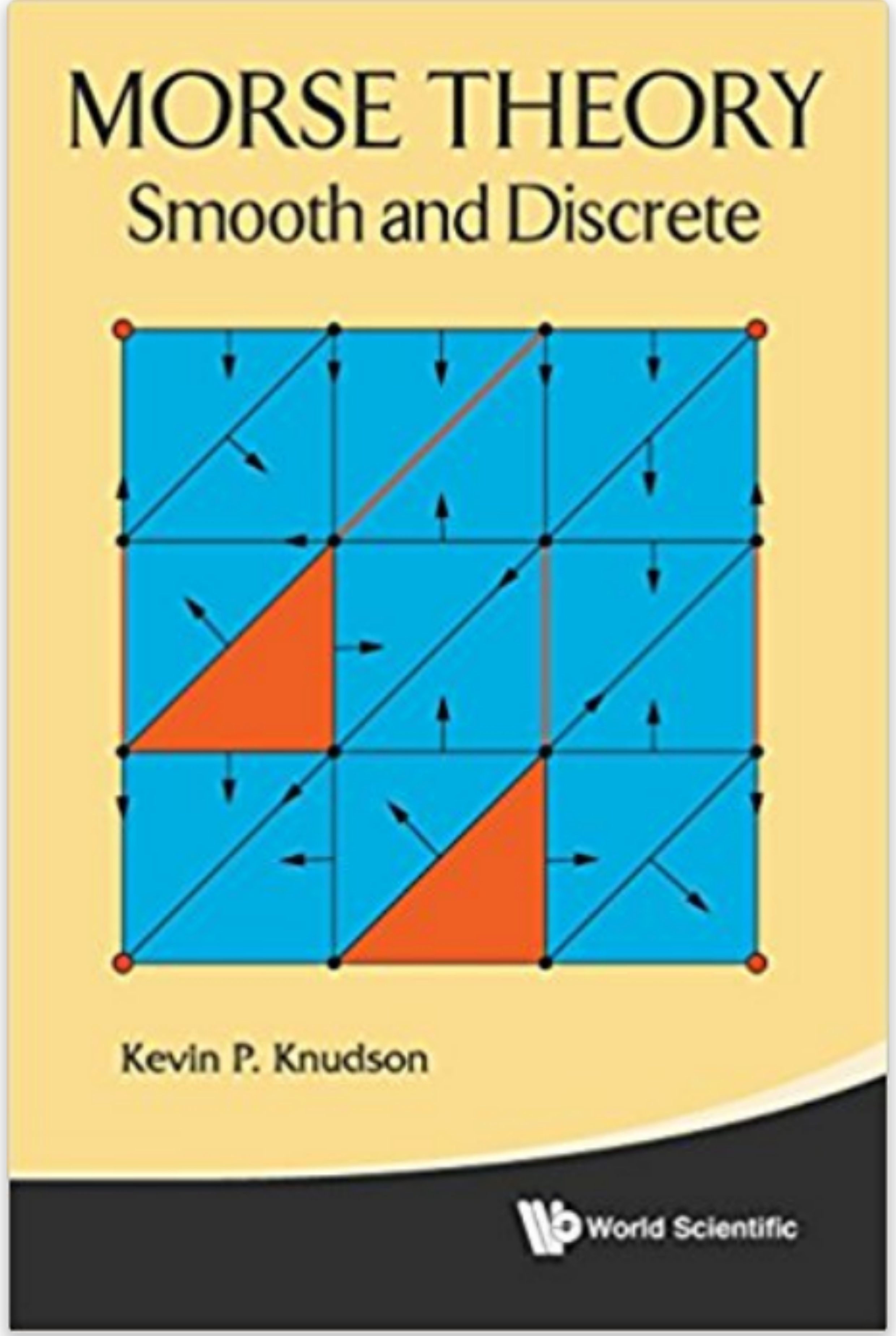


FIGURE 2. A gradient vector field on the torus.

King, Knudson, Mramor 2005




MORSE THEORY
Smooth and Discrete
Kevin P. Knudson
World Scientific

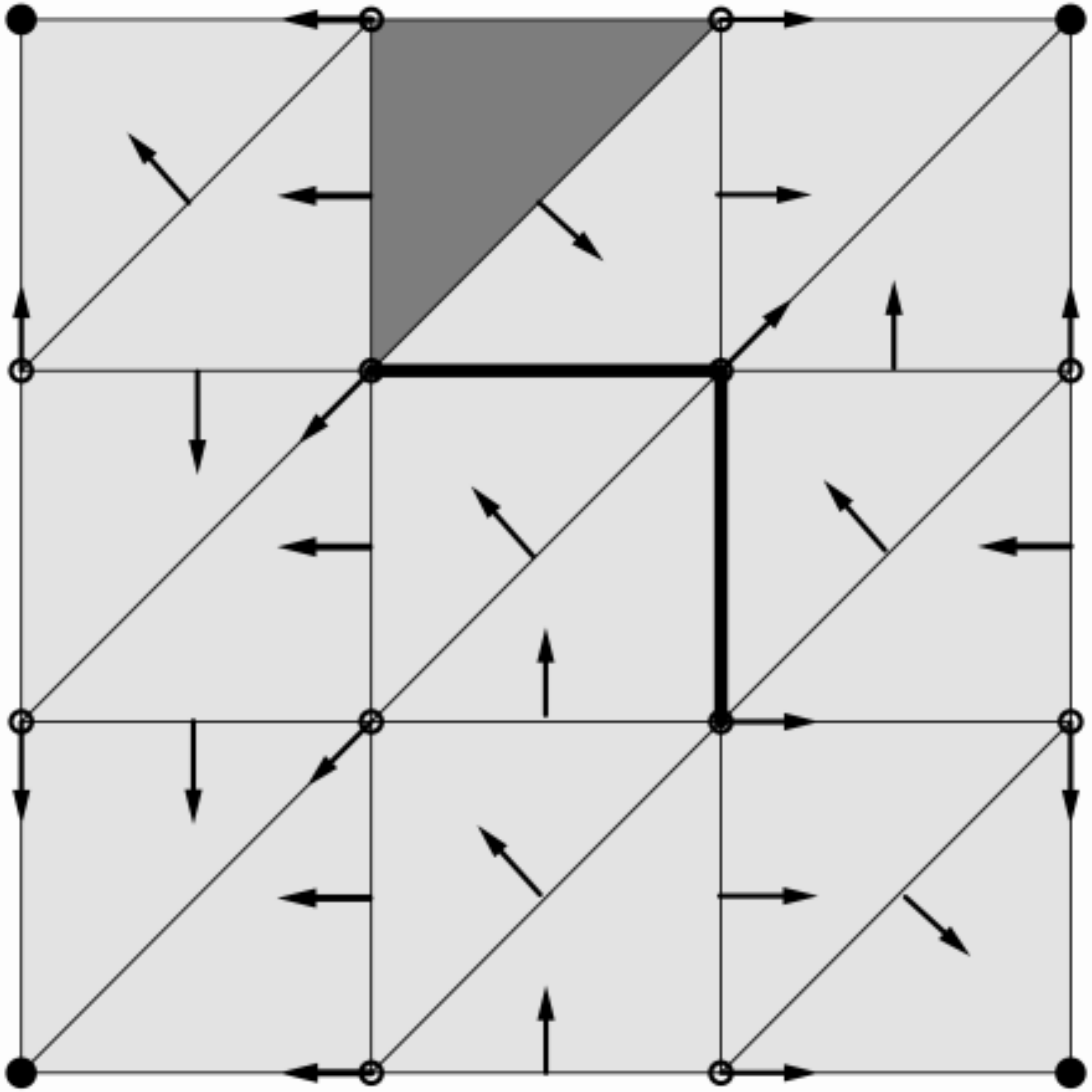


FIGURE 3. A discrete vector field on a triangulated torus. There are four critical cells: the top-center triangle (dark gray), the top and right edges of the center square (indicated with thicker lines), and the vertex obtained by identifying the four corners.

https://arxiv.org/pdf/0808.0051.pdf

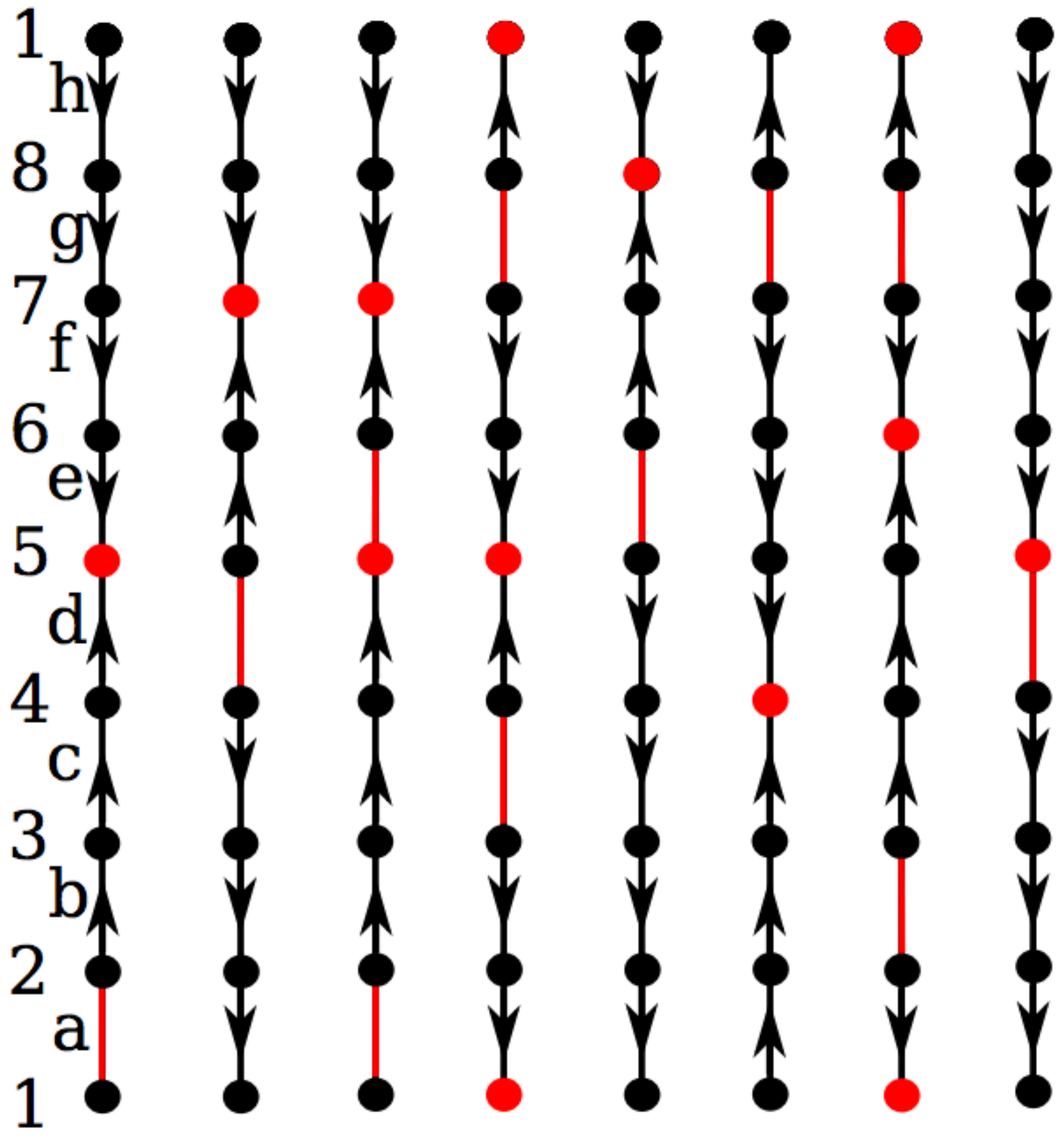


FIGURE 4. A family of discrete gradient fields on the circle. The slices are numbered left to right as $0, 1, \ldots, 7$.

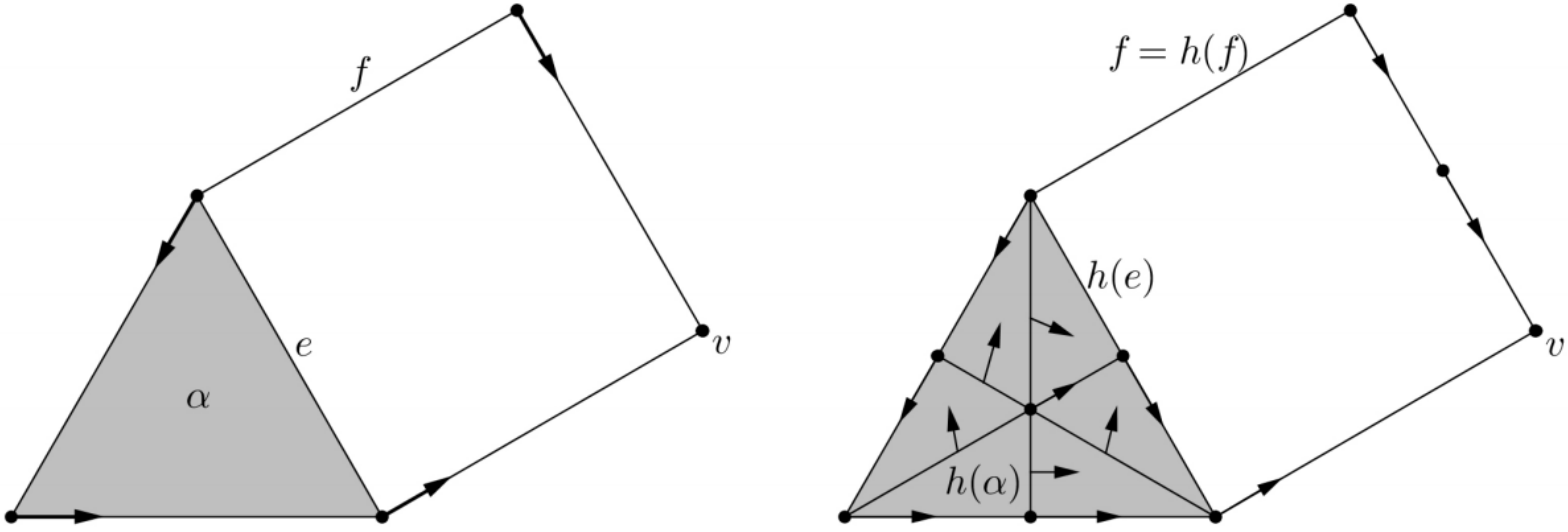
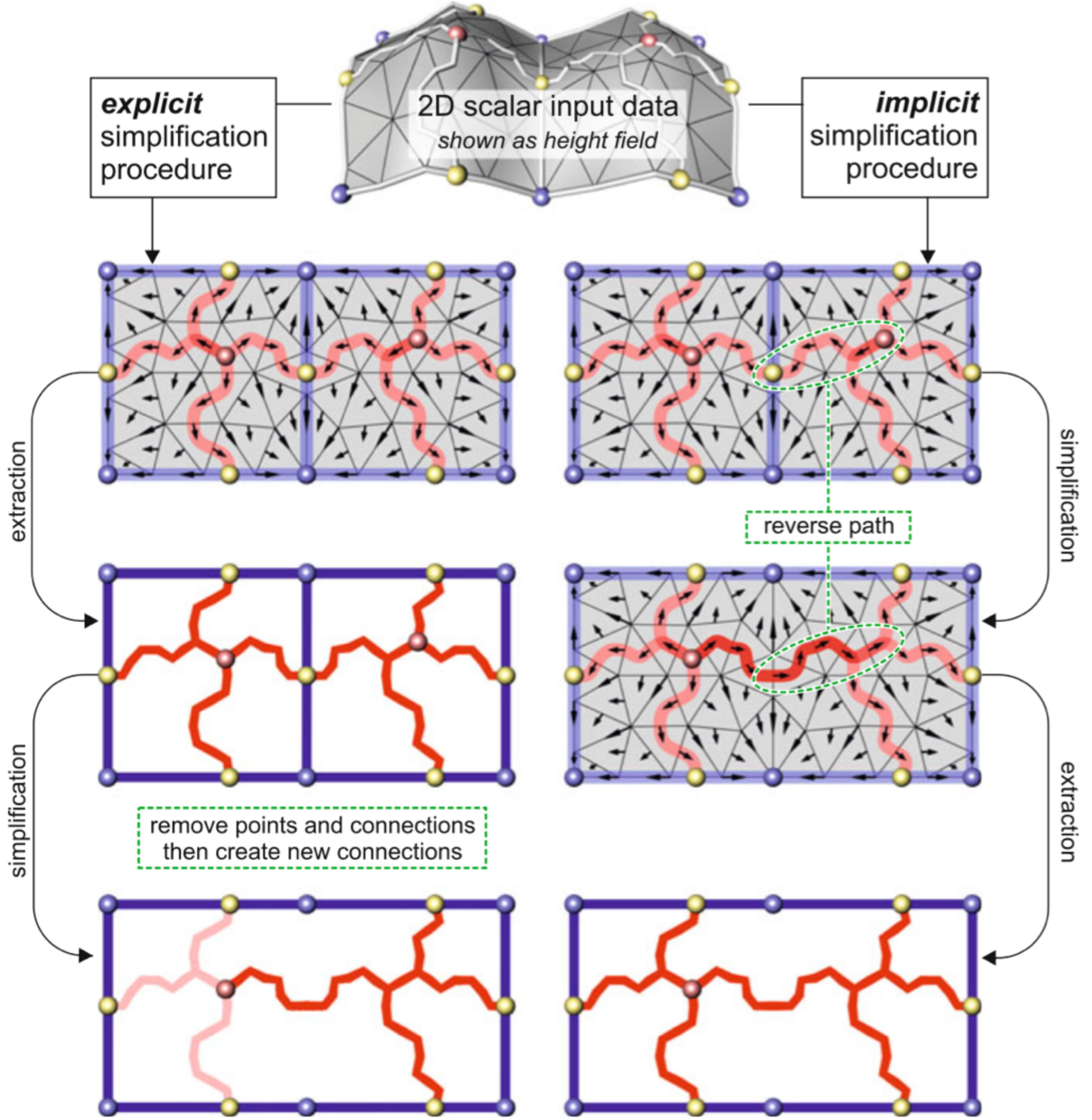


FIGURE 6. Left: A simplicial complex M with a discrete gradient V. The critical cells of V are α, e, f, and v. Right: A refinement N of M and a refinement W of V. The new critical cells $h(\alpha)$ and $h(e)$ are indicated.

Applications

Simplification of MSC



GuntherReininghausSeidel2014

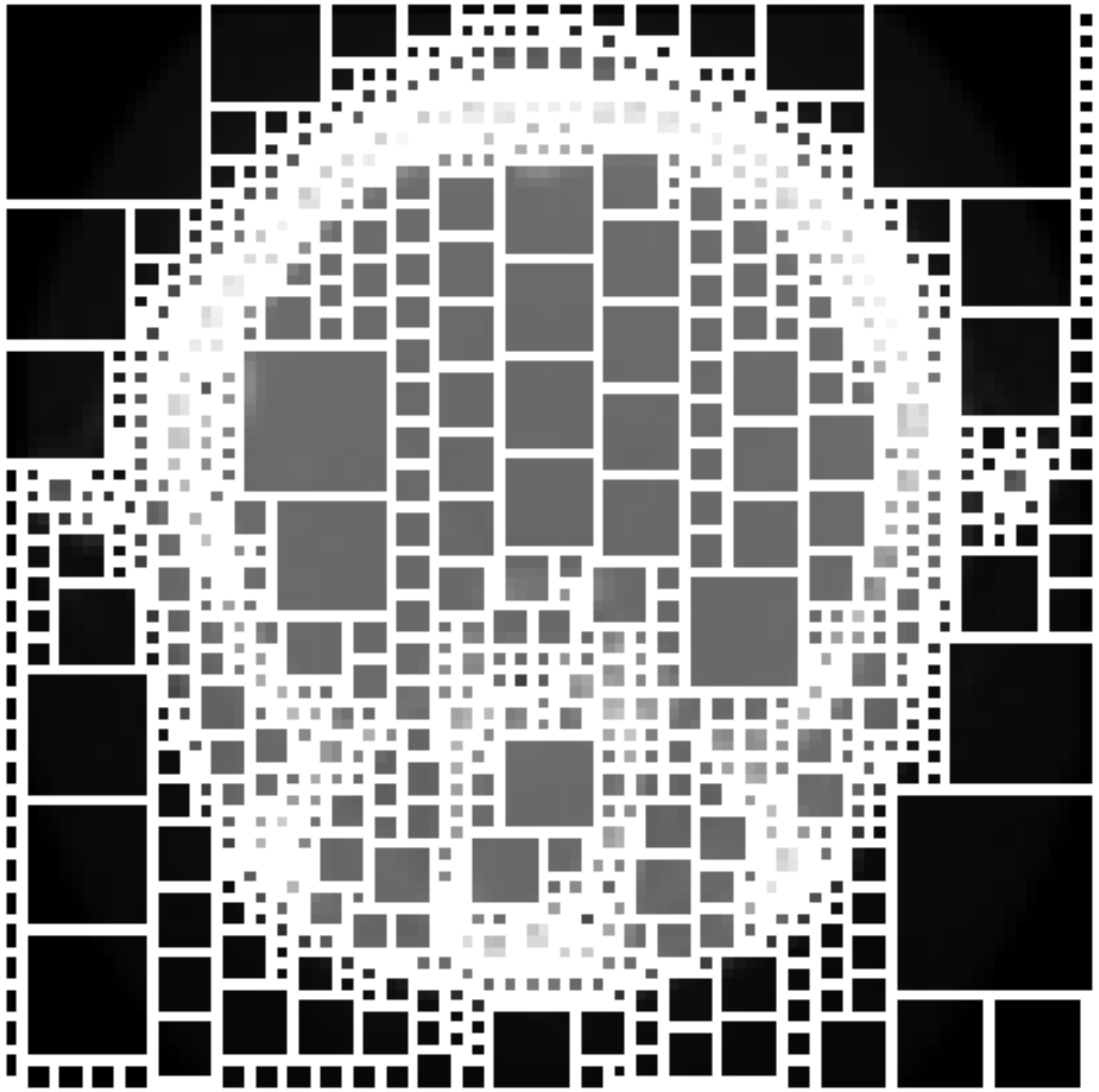


FIGURE 15. Reduced cell decomposition of a head CT scan.

KingKnudsonKosta2016

Thanks!

Any questions?

You can find me at: **beiwang@sci.utah.edu**

CREDITS

Special thanks to all people who made and share these awesome resources for free:

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Colors used